Nonlinear Control Theory

Lecture 1 Introduction

Today:
- Nonlinear vs Linear
- ODE Theory

why nonlinear control?

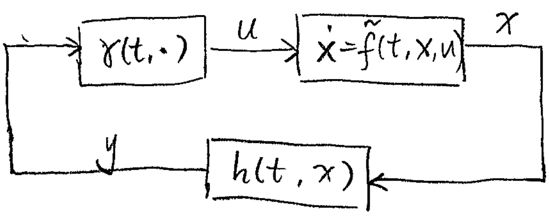
A nonlinear system can be expressed as

$$(*) \quad \begin{aligned} \dot{x} &= \tilde{f}(t, x, u) \leftarrow \text{state equation / dynamics} \\ y &= h(t, x) \end{aligned}$$

in general.

$$\boxed{\begin{aligned} &\text{State: } x \in \mathbb{R}^n \\ &\text{control input: } u \in \mathbb{R}^m \\ &\text{system output: } y \in \mathbb{R}^p \end{aligned}}$$

Goal: design a function $u = \gamma(t, y)$, such that x or y would behave



After designing $u = \gamma(t, x)$, $(*)$ would be:

$$(**) \quad \begin{aligned} \dot{x} &= f(t, x) := \tilde{f}(t, \gamma(t, y)) = \tilde{f}(t, \gamma(t, h(t, x))) \\ y &= h(t, x) \end{aligned}$$

We would mainly analyze the behavior of $(**)$, to make sure our control design $u = \gamma(t, y)$ is "reasonable" (making $x(t)$, or $y(t)$ "behave")

A special case:

$$\dot{x} = f(x) \leftarrow \text{function } f \text{ does not depend explicitly on } t.$$

is called autonomous / time-invariant system.

Ex.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{\gamma}{t} x_2 - x_1^3 \end{aligned}$$

non-autonomous

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\alpha \sin x_1 - \beta x_2 \end{aligned}$$

autonomous.

We have studied linear systems.

$$\dot{x} = A(t)x + B(t)u$$
$$y = C(t)x + D(t)u$$

It is a special case of nonlinear system.

$$\dot{x} = f(x) + g(x)u \approx \underbrace{\frac{\partial f}{\partial x}\Big|_{x=x_0}}_{A} x + \underbrace{\frac{\partial g}{\partial x}\Big|_{x=x_0}}_{B} u.$$

<u>Nice properties</u>:

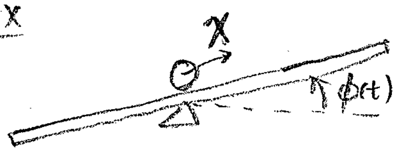
① Local stability = global stability.

② Superposition : Enough to know a step (or impulse) response.

③ Frequency analysis possible : Sinusoidal inputs gives sinusoidal outputs.

<u>But linear models may be too crude !!</u>

<u>Ex</u>



$m\ddot{x} = mg\sin\phi(t)$ Nonlinear model.

$\ddot{x} = g\phi(t)$ Linear model.

Q: can the ball move 0.1 meter in 0.1 seconds from steady state?

$$0.1 = x(0.1) - \underset{=0}{\underset{\uparrow}{x(0)}} = \frac{1}{2}\underset{\approx 10}{g}(0.1)^2\phi_0 \approx 0.05\phi_0 \Rightarrow \phi_0 \approx \frac{0.1}{0.05} = 2 \text{ rad}$$
$$= 114^\circ$$

unrealistic !!

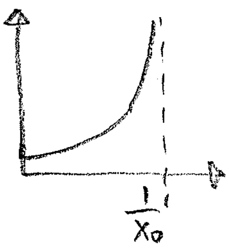
Linear model is only valid for $\sin\phi \approx \phi$.

We need to consider nonlinear models!

<u>Nonlinear models would also have phenomena that linear models does not have such as</u>

1. Finite escape time : $\dot{x} = x^2$, $x(0) = x_0 \Rightarrow x(t) = \frac{x_0}{1 - x_0 t}$.

$0 \le t < \frac{1}{x_0}$

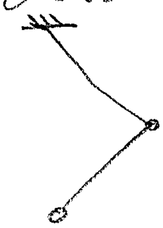


2. Multiple isolated equilibria.

$x = k\pi$

3. Limit circles. (stable, intrinsic oscillations)

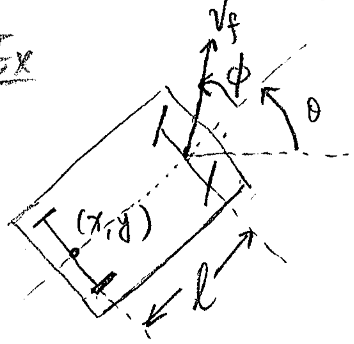
4 Chaos



$\boxed{\text{etc}}$

A motivating nonlinear systems.

Ex



Nonholonomic constraints:

no movement in the direction orthogonal to front wheels. and the rear.

rear: $\dot{x}\sin\theta - \dot{y}\cos\theta = 0$

front: $\frac{d}{dt}(x + l\cos\theta)\sin(\theta+\phi) - \frac{d}{dt}(y + l\sin\theta)\cos(\theta+\phi) = 0$

One can verify $\begin{cases} \dot{x} = v\cos\theta \\ \dot{y} = v\sin\theta \\ \dot{\theta} = \frac{v}{l}\tan\phi \end{cases}$ satisfies nonholonomic constraints.

$\leftarrow$ (Hint: use $l\dot{\theta} = v_f\sin\phi$, $v_f\cos\phi = v$)

We can simplify the model by introducing a new control $w =$

$$\left.\begin{aligned} \dot{x} &= v\cos\theta \\ \dot{y} &= v\sin\theta \\ \dot{\theta} &= w. \end{aligned}\right\} \text{ unicycle model}$$

can rotate where it is even when $v = 0$!

ODE Theory

Consider a nonlinear system

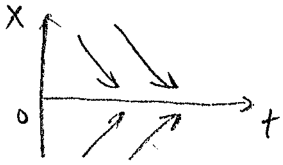
$$\dot{x} = f(t, x), \quad x(t_0) = x_0, \quad t \geq t_0. \ x(t) \in \mathbb{R}^n$$

$$f: [t_0, t_1] \times \underset{\mathbb{R}^n}{\underset{\cap}{\Omega}} \mapsto \mathbb{R}^n.$$

Q:
- Existence of sol.
- Uniqueness of sol.
- How does sol. depend on initial condition?
- How does the sol depend on f?

Ex 1. (Existence)

$$\dot{x} = f(t,x) = \begin{cases} -1, & \text{if } x \geq 0 \\ 1, & \text{if } x < 0 \end{cases}$$



Suppose $x(0) = x_0 = 0$, $x(t)$ is continuously differentiable

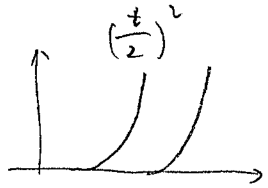
$\Rightarrow \dot{x}(0) = f(0,0) = -1$, so there exist an $\varepsilon > 0$, s.t. $\dot{x}(t) < 0$ for $t \in (0, \varepsilon)$

$\Rightarrow x(t) - x(0) = \int_0^t \underbrace{\dot{x}(\tau)}_{<0} d\tau < 0 \Rightarrow x(t) < 0 \quad \forall t \in (0, \varepsilon)$ contradiction!

$\Rightarrow \dot{x}(t) = 1$ in $(0, \varepsilon)$

Ex 2 (Uniqueness)

$\dot{x} = |x(t)|^{\frac{1}{2}}$, $x(0) = 0$



One solution is $x(t) \equiv 0$

Another solution is $x(t) = (\frac{t}{2})^2$ $\quad \dot{x}(t) = \frac{1}{2}t = |(\frac{t}{2})^2|^{\frac{1}{2}} = |x(t)|^{\frac{1}{2}}$ for $t > 0$.

No unique solution

Ex 3 Finite escape time.

A solution even if it exists locally, it may not exist for all times.

$\dot{x}(t) = x^2$, $t \geq 0$

In order to develop general theory for nonlinear systems, it would be the best if we can say:

"If f satisfies $(\cdots)$, then there exists a unique solution to $\dot{x} = f(t,x)$"

Theorem: (Global existence & uniqueness)

Suppose that $f: [t_0, t_1] \times \mathbb{R}^n \to \mathbb{R}^n$ is

1) piecewise continuous in t

2) "Lipschitz" in x,

Then there exists a unique solution to $\quad \dot{x} = f(t,x)$, $x(t_0) = x_0$

in the interval $[t_0, t_1]$

What is "Lipschitz"?

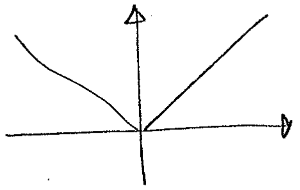
Def: $f(x)$ is Lipschitz if $\exists L > 0$, s.t.

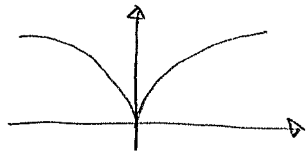
$$\|f(x) - f(y)\| \leq \underset{\substack{\uparrow \\ \text{Lipschitz constant}}}{L} \|x - y\|, \quad \forall x, y.$$

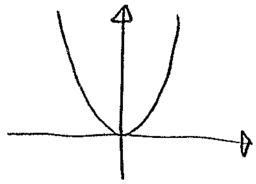
Lipschitz $\Rightarrow$ continuity

Ex

$\lvert x \rvert$	$\sqrt{\lvert x \rvert}$	x^2







$\big| |x| - |y| \big| \leq 1 \cdot |x - y|$

$L = 1$

Not Lipschitz

$|\sqrt{x} - \sqrt{0}| \leq L |x - 0|$

$|\sqrt{x}| \leq L |x|$

$\frac{1}{\sqrt{x}} \leq L$

you can not find L.

Not Lipschitz

$|x^2 - y^2| \leq L |x - y| \quad \forall x, y.$

$y = 0$

$\Downarrow$

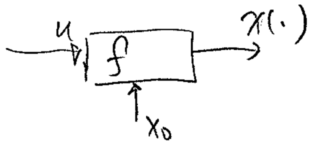
$x^2 \leq L |x|$

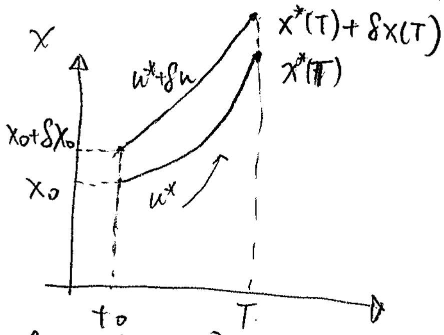
$|x| \leq L$

$\Rightarrow L \to \infty$ as $|x| \to \infty$

Sensitivity to perturbations

$\dot{x}(t) = f(t, x(t), u(t))$, $t \in [0, T]$





$x_0 \to x_0 + \delta x_0$

$u^* \to u^* + \delta u$

How does this affect the trajectory?

Linear analysis

Let $\delta x = x - x^*$.

$$\dot{\delta x} = \dot{x} - \dot{x}^* = f(t, x^* + \delta x, u^* + \delta u) - f(t, x^*, u^*)$$

$$\approx \underbrace{\frac{\partial f}{\partial x}(t, x^*, u^*)}_{A(t)} \delta x + \underbrace{\frac{\partial f}{\partial u}(t, x^*, u^*)}_{B(t)} \delta u$$

$$\Rightarrow \dot{\delta x} = A(t) \delta x + B(t) \delta u, \quad \delta x(0) = \delta x_0$$

δx_0, δu small $\Rightarrow$ δx is small.

How to prove this formally?

Lemma (Grönvall inequality)

If $\varphi(t) \leq \alpha(t) + \int_0^t \beta \varphi(s) ds$, $\beta > 0$,

Then $\varphi(t) \leq \alpha(t) + \int_0^t \beta \alpha(s) e^{\beta(t-s)} ds$

proof : Let $F(t) = \int_0^t \beta \varphi(s) ds$, $F(0) = 0$

$$\Rightarrow \dot{F}(t) = \beta \cdot \varphi(t) \leq \beta \cdot \alpha(t) + \beta \underbrace{\int_0^t \beta \varphi(s) ds}_{F(t)} = \beta \cdot \alpha(t) + \beta F(t)$$

$$e^{-\beta t}(\dot{F}(t) - \beta F(t)) \leq \beta \alpha(t) \cdot e^{-\beta t}$$

$$\frac{d}{dt}(F(t) e^{-\beta t}) \leq \beta \alpha(t) \cdot e^{-\beta t}$$

$\Rightarrow$ Integrate from 0 to t,

$$F(s) e^{-\beta s}\Big|_0^t \leq \int_0^t \beta \alpha(s) e^{-\beta s} ds$$

$$F(t) e^{-\beta t} \leq \int_0^t \beta \alpha(s) e^{-\beta s} ds \Rightarrow F(t) \leq \int_0^t \beta \alpha(s) e^{\beta(t-s)} ds$$

$$\Rightarrow \varphi(t) \leq \alpha(t) + \underbrace{\int_0^t \beta \varphi(s) dt}_{F(t)} \leq \alpha(t) + \int_0^t \beta \alpha(s) e^{\beta(t-s)} ds$$

Thm Suppose $f : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is

1) Piecewise continuous in t
2) Lipschitz continuous in (x, u), i.e.,

$\exists L > 0$, s.t. $\|f(t, x, u) - f(t, y, v)\| \leq L(\|x - y\| + \|u - v\|)$, $\forall x, y \in \mathbb{R}^n$, $u, v \in \mathbb{R}^m$

Let u^* be piecewise continuous function and

Let x^* satisfy $\dot{x}^* = f(t, x^*, u^*)$, $x^*(0) = x_0^*$

Let u be piecewise continuous function, s.t.

$\|u(t) - u^*(t)\| \leq M$, $\forall t \in [0, T]$

and let x be the corresponding trajectory to.

$\dot{x}(t) = f(t, x(t), u(t))$

$x(0) = x_0$.

Then $\|x(t) - x^*(t)\| \leq e^{Lt} \|x_0 - x_0^*\| + M(e^{Lt} - 1)$

$\underline{\text{proof}}$: $x^*(t) = x_0^* + \int_0^t f(s, x^*(s), u^*(s))ds$, $x(t) = x_0 + \int_0^t f(s, x(s), u(s))ds$.

$$\underbrace{\| x^*(t) - x(t) \|}_{\varphi(t)} \le \|x_0^* - x_0\| + \int_0^t \|f(s, x^*(s), u^*(s)) - f(s, x(s), u(s))\| ds$$

$$\le \|x_0^* - x_0\| + \int_0^t L(\|x^*(s) - x(s)\| + \underbrace{\|u^*(s) - u(s)\|}_{\le M}) ds$$

$$\le \underbrace{\|x_0^* - x_0\| + t \cdot LM}_{\alpha(t)} + \int_0^t \underbrace{L}_{\beta} \underbrace{\|x^*(s) - x(s)\|}_{\varphi(s)} ds$$

Use Grönwall inequality

$$\Rightarrow \|x^*(t) - x(t)\| \le \|x_0^* - x_0\| + t \cdot LM + \int_0^t L(\|x_0^* - x_0\| + sLM) e^{L(t-s)} ds$$

$$\Rightarrow \|x^*(t) - x(t)\| \le \|x_0^* - x_0\| (1 + e^{Lt} - 1) + tLM + L^2 M \left(-\frac{t}{L} + \frac{1}{L^2}(e^{Lt} - 1)\right)$$

$$= \|x_0^* - x_0\| e^{Lt} + M(e^{Lt} - 1)$$