Nonlinear Control Theory

Lecture 10. Linearization of Nonlinear Systems II.

Last time:

- The necessary and sufficient condition of equivalence between nonlinear & linear systems.

① $\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i \iff \dot{z} = Az + \sum_{i=1}^{m} b_i u_i$ (A, B) is controllable.

i) $\dim \{ ad_f^k g_i(x_0) \mid 1 \le i \le m,\ 0 \le k \le n-1 \} = n$

ii) there exists a neighbourhood U of x_0, such that $[ad_f^s g_i, ad_f^t g_j] = 0$

② $\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i$, $y = h(x)$ $\iff$ $\dot{z} = Az + \sum_{i=1}^{m} b_i u_i$, $y = Cz$ (A, B, C) is minimal

i) $\dim \{ ad_f^k g_i(x_0) \mid 1 \le i \le m,\ 0 \le k \le n-1 \} = n.$

ii) $\dim \{ d L_f^k h_j(x_0) \mid 1 \le j \le p,\ 0 \le k \le n-1 \} = n.$ consequence of having relative degree n.

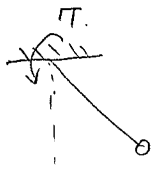
iii) there exists a neighbourhood U of x_0, s.t $L_{g_i} L_f^s L_{g_j}^t h(x) = 0$, $x \in U$, $1 \le i, j \le m$, $s, t \ge 0$.

The transform is: $z = T(x) = \gamma_{z_n}^{X_n} \circ \cdots \circ \gamma_{z_1}^{X_1}(x_0)$, $X_i = ad_f^k g_j$.

Today.

- Feedback linearization. (SISO systems)

Ex) Stabilizing a pendulum at $\theta = \delta$



$\ddot{\theta} = -a \sin\theta - b\dot{\theta} + cT$

Steady torque $0 = -a \sin\delta + cT_s$

State $x_1 = \theta$, $x_2 = \dot{\theta}$ control input $u = T - T_s$

$\Rightarrow \dot{x}_1 = x_2$

$\dot{x}_2 = -(a \sin(x_1 + \delta) - \sin\delta) - bx_2 + cu$ where the "true" control happens.

Suppose $c \neq 0$, let $u = \underbrace{\frac{a}{c}[\sin(x_1 + \delta) - \sin\delta]}_{\text{is used to cancel the nonlinear terms.}} + \frac{v}{c}$

$\Rightarrow \dot{x}_1 = x_2$

$\dot{x}_2 = -bx_2 + v$

Idea: Cancel the "nonlinear term" with the control input

Question: are we always able to do such tricks?

Before we proceed, we will cover a theorem that plays a fundamental role in nonlinear systems theory, which is called <u>Frobenius theorem</u>.

> Recall the definition of "<u>distribution</u>" Δ, it is a map that assigns to each $p \in M$ a vector space $\Delta(p)$ of the tangent space T_pM. Δ is smooth if for each $p \in M$, there exists a neighbourhood U of p and a set of <u>smooth</u> vector fields $\{f_i\}$, $i \in I$, such that $\Delta(q) = \text{span}\{f_i(q), i \in I\}$, $\forall q \in U$..

Now, consider a smooth distribution $\Delta(x) = \text{span}\{f_1(x), \cdots, f_d(x)\}$, which is <u>nonsingular</u>. $\forall x \in \underbrace{U(x_0)}_{\text{neighbourhood of } x_0}$, $x_0 \in U \subseteq \mathbb{R}^n$. If $\omega_i \in \underbrace{T_x^*M}_{\text{cotangent space}}$, namely, ω_i are co-vector fields,

We can construct co-distribution similarly as $\Omega = \text{span}\{\omega_1, \cdots, \omega_l\}$.

Moreover, if we construct the co-vector field as: $\langle \omega_j(x), f_i(x) \rangle = 0$, $i = 1, \cdots d$, $j = 1, \cdots, n-d$

We can construct the co-distribution $\Omega = \Delta^\perp$, that has dimention $n-d$.
annihilator

This actually means $\omega_j(x) \underbrace{F(x)}_{[f_1, \cdots, f_d]} = 0$

Now, suppose $\omega_j(x)$ are <u>exact one form</u>, namely, $\omega_j(x) = d\lambda_j(x)$

in local coordinates $\frac{\partial \lambda_j(x)}{\partial x}$.

$$\frac{\partial \lambda_j}{\partial x} \cdot F(x) = 0, \quad j = 1, \cdots, n-d.$$

<u>Question</u> when does the solutions for this set of differential equations exists?

$\Leftrightarrow$ when does a nonsingular distribution Δ has <u>an annihilator $\Delta^\perp$</u>, which is <u>spanned by</u> <u>exact one-form</u>?

"completely integrable"

<u>Def</u> (<u>Completely integrable</u>)

A nonsingular d-dimentional distribution Δ, defined on an open set U of $\mathbb{R}^n$, is said to be <u>completely integrable</u> if $\forall x^0 \in U$, there $\exists U^0(x^0)$, and $\lambda_1, \cdots, \lambda_{n-d} \in C^\infty(U^0)$, such that $\text{span}\{d\lambda_1, \cdots, d\lambda_{n-d}\} = \Delta^\perp$.

<u>Thm</u> (Frobenius)

A nonsingular distribution is <u>completely integrable</u> iff it is <u>involutive</u>.

$f_1 \in \Delta$, $f_2 \in \Delta$
$\Rightarrow [f_1, f_2] \in \Delta$

Consider SISO system

$$(*)\quad \begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned}, \quad x \in \mathbb{R}^n, y \in \mathbb{R}, u \in \mathbb{R}.$$

The analysis starts from the concept of "Relative Degree"

Def. (Relative Degree)

The system $(*)$ is said to have relative degree r at x° if

i) $L_g L_f^k h(x) = 0, \ \forall x \in U(x^\circ), \ \forall k < r-1$

ii) $L_g L_f^{r-1} h(x^\circ) \neq 0$

Ex | $\begin{aligned} \dot{x} &= Ax + bu \\ y &= Cx \end{aligned} \Rightarrow \begin{aligned} L_f^k h(x) &= CA^k x \\ L_g L_f^k h(x) &= CA^k b \end{aligned} \Rightarrow$ If the linear system has relative degree r then $CA^k b = 0 \ \forall k < r-1$, $CA^{r-1} b \neq 0$

$\Rightarrow$ Transfer function $G(s) = C(sI - A)^{-1} b$

$$= CBs^{-1} + CAbs^{-2} + \cdots$$

$\Rightarrow$ Relative degree is the difference between the degree of denominator polynomial & nominator polynomial of the transfer function.

(An interpretation)

Ex | Suppose $x(t^\circ) = x^\circ$,

$$y(t^\circ) = h(x(t^\circ)) = h(x^\circ)$$

$$y^{(1)}(t) = \frac{\partial h}{\partial x} \cdot (f(x) + g(x)u) = L_f h(x(t)) + L_g h(x(t)) u(t)$$

- If relative degree $r > 1$, $L_g h(x(t)) = 0$, $\forall x$ near $x^\circ \Rightarrow y^{(1)}(t) = L_f h(x(t))$

This yields $y^{(2)}(t) = \dfrac{\partial L_f h(x)}{\partial x} (f(x) + g(x) \cdot u) = L_f^2 h(x(t)) + L_g L_f h(x(t)) u(t)$

- If relative degree $r > 2$, $L_g L_f h(x(t)) = 0$, $\forall x$ near $x^\circ \Rightarrow y^{(2)}(t) = L_f^2 h(x(t))$

$\vdots$

$$y^{(k)}(t) = L_f^k h(x(t)), \quad \forall k < r, \ \forall t \text{ near } t^\circ$$

$$y^{(r)}(t) = L_f^r h(x^\circ) + L_g L_f^{r-1} h(x^\circ) u(t^\circ)$$

Relative degree r is exactly the number of times one has to differentiate the output $y(t)$ at $t = t^\circ$ to have the control input $u(t^\circ)$ show up.

If the relative degree r can not be defined, it means that

$$y(t) = \sum_{k=0}^{\infty} y^{(k)}(t^\circ) \frac{(t - t^\circ)^k}{k!}$$

the output $y(t)$ does not depend on $u(t)$!

$$= L_f^k h(x^\circ)$$

Lemma 1 Let ϕ be a real-valued function and f, g are vector fields, all defined in an open set $U \subseteq \mathbb{R}^n$. Then for $\forall s, k, r \geq 0$, it holds that

$$\underbrace{\langle dL_f^s \phi(x), ad_f^{k+r} g(x) \rangle}_{L_{ad_f^{k+r} g} L_f^s \phi} = \sum_{i=0}^{r} (-1)^i \binom{r}{i} L_f^{r-i} \underbrace{\langle dL_f^{s+i} \phi(x), ad_f^k g(x) \rangle}_{L_{ad_f^k g} L_f^{s+i} \phi}.$$

As a consequence, the following are equivalent:

i) $L_g \phi(x) = L_g L_f \phi(x) = \cdots = L_g L_f^k \phi(x) = 0, \ \forall x \in U$

ii) $L_g \phi(x) = L_{ad_f g} \phi(x) = \cdots = L_{ad_f^k g} \phi(x) = 0, \ \forall x \in U$

i) $\Rightarrow$ ii) : By letting $s=0$ $k=0$ $\quad L_{ad_f^r} \phi = \sum_{i=0}^{r} \cdots \underbrace{L_g L_f^i \phi}_{0}$

ii) $\Rightarrow$ i) $\quad L_g \phi(x) = 0, \ L_{ad_f g} \phi(x) = 0 \Rightarrow L_f \underbrace{L_g \phi}_{0} - L_g L_f \phi = 0 \Rightarrow L_g L_f \phi = 0.$

$$L_{ad_f^2 g} \phi = L_f \underbrace{L_{ad_f g} \phi}_{=0} - L_{ad_f g} L_f \phi = -L_f^2 \phi + \underbrace{L_g L_f \phi}_{=0} = 0 \Rightarrow L_g L_f^2 \phi = 0$$

$\cdots$

Lemma 2 The row vectors $dh(x^0), dL_f h(x^0), \cdots, dL_f^{r-1} h(x^0)$ are linearly independent

proof Use this equation, we have

$$\langle dL_f^j h(x), ad_f^i g(x) \rangle = \sum_{l=0}^{i} (-1)^l \binom{i}{l} L_f^{i-l} L_g L_f^{j+l} h(x) \quad \rightarrow \max\ j+i$$

$$\Rightarrow \begin{cases} \langle dL_f^j h(x), ad_f^i g(x) \rangle = 0, & \text{if } i+j \leq r-2, \ \forall x \text{ around } x^0 \\ \langle dL_f^j h(x), ad_f^i g(x) \rangle = (-1)^i L_g L_f^{r-1} h(x^0) \neq 0, & i+j = r-1 \end{cases}$$

$$\Rightarrow \underbrace{\begin{bmatrix} dh(x^0) \\ dL_f h(x^0) \\ \vdots \\ dL_f^{r-1} h(x^0) \end{bmatrix}}_{r \times n} \underbrace{[\, g(x^0) \ \ ad_f g(x^0) \ \cdots \ ad_f^{r-1} g(x^0) \,]}_{n \times r} = \underbrace{\begin{bmatrix} 0 & \cdots & 0 & \cdots & \langle dh(x^0), ad_f^{r-1} g(x^0) \rangle \\ 0 & & & & * \\ \vdots & & & \langle dL_f^{r-2} h(x^0), ad_f g(x^0) \rangle & * \\ 0 & & & & \vdots \\ \langle dL_f^{r-1} h(x^0), g(x^0) \rangle & * & & & * \end{bmatrix}}_{\text{rank} = r}$$

$\Rightarrow$ $dh(x^0), \cdots, dL_f^{r-1} h(x^0)$ are linearly independent

$g(x^0), \cdots, ad_f^{r-1} g(x^0)$ are —— || ——

Feedback linearization

Consider an SISO system. $\dot{x} = f(x) + g(x) u$, given a point x^0, find a neighbourhood $U(x^0)$, a feedback $u = \alpha(x) + \beta(x) v$ defined on $U(x^0)$, and a coordinate change $z = \phi(x)$ defined on $U(x^0)$, such that $\dot{x} = f(x) + g(x)\alpha(x) + g(x)\beta(x) v$ expressed in the z-coordinates, is linear and controllable.

$$\dot{z} = Az + Bv, \quad (A, B) \text{ controllable}$$

Lemma 3 The above problem is solvable iff there exists a neighbourhood $U(x^\circ)$ and a real-valued function $\lambda(x)$, defined on $U(x^\circ)$, such that the system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= \lambda(x) \end{aligned}$$ has relative degree n at x°.

proof (Sufficiency)

Suppose there exists $\lambda(x)$, such that the system has a relative degree n

Namely, $L_g\lambda(x)=0,\ L_gL_f\lambda(x)=0,\ \dots,\ L_gL_f^{n-2}\lambda(x)=0,\ L_gL_f^{n-1}\lambda(x)\neq 0.$

$$\Rightarrow\ z_1=\phi_1(x)=\lambda(x)\ \Rightarrow\ \dot z_1=\frac{\partial\phi_1}{\partial x}\cdot(f(x)+g(x)\cdot u)=\frac{\partial\lambda(x)}{\partial x}f(x)+\frac{\partial\lambda(x)}{\partial x}g(x)\cdot u$$

$$=L_f\lambda(x)+\underbrace{L_g\lambda(x)}_{=0}\cdot u$$

$$z_2=\phi_2(x)=L_f\lambda(x)\ \Rightarrow\ \dot z_2=\frac{\partial\phi_2}{\partial x}\cdot(f(x)+g(x)\cdot u)=\frac{\partial L_f\lambda(x)}{\partial x}\cdot f(x)+\frac{\partial L_f\lambda(x)}{\partial x}g(x)u$$

$$=L_f^2\lambda(x)+\underbrace{L_gL_f\lambda(x)}_{=0}\cdot u$$

$$\vdots$$

$$z_{n-1}=\phi_{n-1}(x)=L_f^{n-2}\lambda(x)\ \Rightarrow\ \dot z_{n-1}=\frac{\partial\phi_{n-1}}{\partial x}(f(x)+g(x)u)=\frac{\partial L_f^{n-2}\lambda(x)}{\partial x}f(x)+\frac{\partial L_f^{n-2}\lambda(x)}{\partial x}g(x)u$$

$$=L_f^{n-1}\lambda(x)+\underbrace{L_gL_f^{n-2}\lambda(x)}_{=0}u$$

$$z_n=\phi_n(x)=L_f^{n-1}\lambda(x)\ \Rightarrow\ \dot z_n=\frac{\partial L_f^{n-1}\lambda(x)}{\partial x}(f(x)+g(x)u)=L_f^n\lambda(x)+\frac{\partial L_f^{n-1}\lambda(x)}{\partial x}g(x)u$$

$$=L_f^n\lambda(x)+\underbrace{L_gL_f^{n-1}\lambda(x)}_{\neq 0}g(x)u$$

$$\Rightarrow\ u=\underbrace{-\frac{L_f^n\lambda(x)}{L_gL_f^{n-1}\lambda(x)\,g(x)}}_{\alpha(x)}+\underbrace{\frac{1}{L_gL_f^{n-1}\lambda(x)}}_{\beta(x)}v$$

The system becomes $\begin{cases}\dot z_1=z_2\\ \ \vdots\\ \dot z_{n-1}=z_{n-2}\\ \dot z_n=v\end{cases}$ $\Rightarrow$ controllable.

(Necessity)

We first show relative degree does not change with coordinate transformation and feedback.

Let $\bar f(z)=\frac{\partial\phi}{\partial x}f(x)\Big|_{x=\phi^{-1}(z)}$, $\bar g(z)=\frac{\partial\phi}{\partial x}g(x)\Big|_{x=\phi^{-1}(z)}$, $\bar h(z)=h(\phi^{-1}(z))$

$$\text{then } L_{\bar f}\bar h(z)=\frac{\partial\bar h}{\partial z}\bar f(z)=\frac{\partial h}{\partial x}\Big|_{x=\phi^{-1}(z)}\underbrace{\frac{\partial\phi^{-1}}{\partial z}\cdot\frac{\partial\phi}{\partial x}}_{=I}f(x)\Big|_{x=\phi^{-1}(z)}$$

$$=\frac{\partial h}{\partial x}f(x)\Big|_{x=\phi^{-1}(z)}=L_fh(x)\Big|_{x=\phi^{-1}(z)}$$

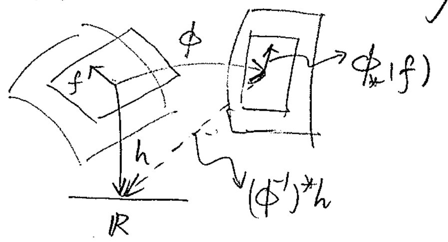
$$L_{\bar g} L_{\bar f} \bar h(z) = \frac{\partial L_f h(x)}{\partial x}\Big|_{x=\phi^{-1}(z)} \underbrace{\frac{\partial \phi^{-1}}{\partial z}\left[\frac{\partial \phi}{\partial x} g(x)\right]}_{=I}\Big|_{x=\phi^{-1}(z)} = L_g L_f h(x)\Big|_{x=\phi^{-1}(z)}$$

$$L_{\bar f}^2 \bar h(z) = \frac{\partial L_f h(x)}{\partial x}\Big|_{x=\phi^{-1}(z)} \underbrace{\frac{\partial \phi^{-1}}{\partial z}\left[\frac{\partial \phi}{\partial x} f(x)\right]}_{=I}\Big|_{x=\phi^{-1}(z)} = L_f^2 h(x)\Big|_{x=\phi^{-1}(z)}$$

$\vdots$

Iterating this, we have $L_{\bar g} L_{\bar f}^k \bar h(z) = [L_g L_f^k h(x)]\big|_{x=\phi^{-1}(z)}$ $\left(L^k_{\phi_*(f)} (\phi^{-1})^* h = (\phi^{-1})^*(L_f^k h) \right)$

Relative degree does not change under coordinate change!



On the other hand, we claim that

$L^k_{f+g\alpha} h(x) = L_f^k h(x)$, $0 \le k \le r-1$.

show this by induction, it is trivially true for $k=0$.

Suppose this is the case for k.

Now $L^{k+1}_{f+g\alpha} h(x) = \frac{\partial L^k_{f+g\alpha} h(x)}{\partial x} (f+g\alpha) = \frac{\partial L_f^k h(x)}{\partial x} f + \frac{\partial L_f^k h(x)}{\partial x} g\alpha$

$= L_f^{k+1} h(x) + \underbrace{L_g L_f^k h(x)}_{=0 \text{ for } k \le r-2} \alpha$

$L_{g\beta} L^k_{f+g\alpha} h(x) = L_{g\beta} L_f^k h(x) = \frac{\partial L_f^k h(x)}{\partial x} g\beta = \underbrace{L_g L_f^k h(x)}_{=0,\ k\le r-2} \beta(x)$

and $L_{g\beta} L^{r-1}_{f+g\alpha} h(x) = L_{g\beta} L_f^{r-1} h(x) = \underbrace{L_g L_f^{r-1} h(x)}_{\neq 0} \cdot \underbrace{\beta(x)}_{\neq 0} \neq 0$.

Relative degree does not change under feedback!

Now let (A, B) be controllable, $\exists$ nonsingular T that transform

$$T(A+BK)T^{-1} = \begin{bmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad TB = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

that is, we design a control $V = kz + V'$

and do a coordinate change to change $\dot z = Az + BV$ into

$$\dot{\bar z} = T(A+BK)T^{-1} \bar z + TBV'$$

we can construct the output as $y = [1\ 0 \cdots 0]\bar z$.

The relative degree for such system is n, and the system is in its controllable canonical form. Since relative degree does not change under coordinate change & feedback, the necessity is proved.

The problem of linearizing the system boils down to finding a function $\lambda(x)$ such that the system has a relative degree n, namely, finding $\lambda(x)$ such that

$$\begin{cases} L_g \lambda(x) = L_g L_f \lambda(x) = \cdots = L_g L_f^{n-2} \lambda(x) = 0 \quad \forall x \in U(x^0) \\ L_g L_f^{n-1} \lambda(x^0) \neq 0 \end{cases} \overset{\text{Lemma 1}}{\Longleftrightarrow} (*) \begin{cases} L_g \lambda(x) = L_{ad_f g} \lambda(x) = \cdots = L_{ad_f^{n-2} g} \lambda(x) = 0 \quad \forall x \in U(x^0) \\ L_{ad_f^{n-1} g} \lambda(x^0) \neq 0 \end{cases}$$

Lemma 4 There exists a real-valued function $\lambda(x)$ defined in $U(x^0)$ solving the PDE (*) iff

i) the matrix $[g(x^0), ad_f g(x^0), \cdots, ad_f^{n-2} g(x^0), ad_f^{n-1} g(x^0)]$ has rank n.

ii) the distribution $\Delta = span\{g, ad_f g, \cdots, ad_f^{n-2} g\}$ is involutive in $U(x^0)$.

proof (Necessity)

Suppose $\lambda(x)$ that solves (*) exists, namely, the system has relative degree n. This satisfies the condition of Lemma 2. From the proof of Lemma 2, we have $[g(x^0), ad_f g(x^0), \cdots, ad_f^{n-1} g(x^0)]$ are linearly independent as well. $\Rightarrow$ i).

Since i) holds, then Δ is nonsingular and is of $n-1$ dimension in $U(x^0)$

In fact, (*) can be written as

$$\begin{array}{ccccc} L_g \lambda(x) = 0, & L_{ad_f g}\lambda(X) = 0, & \cdots & L_{ad_f^{n-2} g}\lambda(x) = 0 \\ \Downarrow & \Downarrow & & \Downarrow \\ \langle d\lambda, g\rangle = 0 & \langle d\lambda, ad_f g\rangle = 0 & & \langle d\lambda, ad_f^{n-2} g\rangle = 0 \end{array}$$

$\Rightarrow$ $d\lambda(x)[g(x), ad_f g(x), \cdots ad_f^{n-2} g] = 0$

$\Rightarrow$ $d\lambda(x)$ is a basis of the 1-dimentional co-distribution $\Delta^\perp$ in $U(x^0)$

$\xRightarrow{\text{Frobenius}}$ Δ is involutive $\Rightarrow$ ii).

(Sufficiency)

i) $\Rightarrow$ $\Delta = span\{g(x^0), ad_f g(x^0), \cdots, ad_f^{n-2} g(x^0)\}$ is nonsingular.

ii) $\xRightarrow{\text{Frobenius}}$ there exists a real-valued function $\lambda(x)$ defined in $U(x^0)$, such that $\Delta^\perp = \underbrace{span\{d\lambda\}}_{\text{solves (*) (the "equals-to-zero" part)}}$

and $L_{ad_f^{n-1} g}\lambda(x^0) \neq 0$ (otherwise $d\lambda$ would not exist since $[g(x^0), ad_f g(x^0), \cdots, ad_f^{n-1} g(x^0)]$ has rank n).

Thm The single input system $\dot{x} = f(x) + g(x)u$ is feedback linearizable at x^0 iff

i) $[g(x^0), ad_f g(x^0), \cdots, ad_f^{n-2} g(x^0), ad_f^{n-1} g(x^0)]$ has rank n.

ii) $\Delta = span\{g, ad_f g, \cdots, ad_f^{n-2} g\}$ is involutive.

single input system

$\Updownarrow$

find $\lambda(x)$ has relative degree (Lemma 3)

$\Updownarrow$

i) ii) in Lemma 4 $\begin{cases} \text{Lemma 2} \Leftarrow \text{Lemma 1} \\ \text{Frobenius Thm} \end{cases}$

SISO system has relative degree n.

$\Rightarrow$ $\dot{x} = f(x) + g(x) \cdot u$

$y = h(x)$

One can immediately linearize it

$\boxed{Ex}$ $\dot{x} = \begin{bmatrix} x_3(1+x_2) \\ x_1 \\ x_2(1+x_1) \end{bmatrix} + \begin{bmatrix} 0 \\ 1+x_2 \\ -x_3 \end{bmatrix} u.$

$$ad_f g(x) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_3(1+x_2) \\ x_1 \\ x_2(1+x_1) \end{bmatrix} - \begin{bmatrix} 0 & x_3 & 1+x_2 \\ 1 & 0 & 0 \\ x_2 & 1+x_1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1+x_2 \\ -x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ x_1 \\ -(1+x_1)(1+2x_2) \end{bmatrix}$$

$$ad_f^2 g(x) = \begin{bmatrix} (1+x_2)(1+2x_2)(1+x_1) - x_3 x_1 \\ x_3(1+x_2) \\ -x_3(1+x_2)(1+2x_2) - 3x_1(1+x_1) \end{bmatrix}$$

$$[g(x) \ \ ad_f g(x) \ \ ad_f^2 g(x)]_{x=0} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

rank 3.
$\Rightarrow$ i) is satisfied.

$$[g, ad_f g] = \begin{bmatrix} 0 \\ * \\ * \end{bmatrix} \Rightarrow \Delta = \text{span}\{g, ad_f g\} \text{ is involutive.}$$

$$\frac{\partial \lambda(x)}{\partial x}[g(x), ad_f g(x)] = \frac{\partial \lambda(x)}{\partial x} \begin{bmatrix} 0 & 0 \\ 1+x_2 & x_1 \\ -x_3 & -(1+x_1)(1+2x_2) \end{bmatrix} \Rightarrow \lambda(x) = x_1$$

Let's check if $\dot{x} = \begin{bmatrix} x_3(1+x_2) \\ x_1 \\ x_2(1+x_1) \end{bmatrix} + \begin{bmatrix} 0 \\ 1+x_2 \\ -x_3 \end{bmatrix} u$ has relative degree 3.

$$y = x_1$$

$L_g \lambda(x) = 0 \quad L_g L_f \lambda(x) = 0, \quad L_g L_f^2 \lambda(x) = (1+x_1)(1+x_2)(1+2x_2) - x_1 x_3$

and $L_g L_f^2 \lambda(0) = 1 \neq 0$

Around $x=0$, the system will be transformed into a linear system by

$$u = -\frac{L_f^3 \lambda(x) + v}{L_g L_f^2 \lambda(x)}, \quad \begin{aligned} z_1 &= \lambda(x) = x_1 \\ z_2 &= L_f \lambda(x) = x_3(1+x_2) \\ z_3 &= L_f^2 \lambda(x) = x_3 x_1 + (1+x_1)(1+x_2)x_2. \end{aligned}$$