Nonlinear Control Theory

Lecture 11. Linearization of Nonlinear Systems III

Last time

- Feedback linearization for SISO systems.
  - Relative degree
  - SISO nonlinear affine system

$\dot{x} = f(x) + g(x)u$ is feedback linearizable at x^0

$\Leftrightarrow$ I can find $\lambda(x)$, such that $\begin{aligned}\dot{x} &= f(x)+g(x)u \\ y &= \lambda(x)\end{aligned}$ has relative degree n

$\Leftrightarrow$ i) $[g(x^0), ad_f g(x^0), \cdots, ad_f^{n-1} g(x^0)]$ has rank n

ii) the distribution $\Delta = \text{span}\{g, ad_f g, \cdots, ad_f^{n-2} g\}$ is involutive in $U(x^0)$

Today

Feedback linearization for MIMO systems.

"square system" same input & output dimension

Consider $\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i = f(x) + g(x) u$, $\quad x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^m,$

$g: \mathbb{R}^n \mapsto \mathbb{R}^{n\times m}, h: \mathbb{R}^n \mapsto \mathbb{R}^m$

$(*)$ $\begin{aligned} y_1 &= h_1(x) \\ &\vdots \\ y_m &= h_m(x)\end{aligned}$ $\Leftrightarrow$ $y = h(x)$

Def (Relative degree for MIMO square system)

The "square system" $(*)$ is said to have a relative degree $(r_1, \cdots, r_m)$ at a point x^0 if

i) $L_{g_j} L_f^k h_i(x) = 0, \ \forall\, 1 \le j \le m, \ \forall\, k < r_i - 1, \ \forall\, 1 \le i \le m, \ \forall\, x \in U(x^0)$.

$U(x^0)$: neighbourhood of x^0

ii) the $m \times m$ matrix

$$A(x) = \begin{bmatrix} L_{g_1} L_f^{r_1-1} h_1(x), & \cdots, & L_{g_m} L_f^{r_1-1} h_1(x) \\ L_{g_1} L_f^{r_2-1} h_2(x), & \cdots, & L_{g_m} L_f^{r_2-1} h_2(x) \\ \cdots & \cdots & \cdots \\ L_{g_1} L_f^{r_m-1} h_m(x), & \cdots, & L_{g_m} L_f^{r_m-1} h_m(x) \end{bmatrix}$$

is nonsingular at $x = x^0$

Intepretation

① Extension of SISO case. At least one choice of j such that y_i — u_j SISO system has relative degree r_i.

② r_i is exactly the number of times one has to differentiate the i-th output $y_i(t)$ at $t = t^0$ to have at least one component of $u(t^0)$ show-up.

Lemma 1 Suppose the "square system" (*) has a relative degree $(r_1, \cdots r_m)$ at x^0.

Then the row vectors:

$$\begin{aligned} &dh_1(x^0), dL_f h_1(x^0), \cdots, dL_f^{r_1-1} h_1(x^0), \\ &dh_2(x^0), dL_f h_2(x^0), \cdots, dL_f^{r_2-1} h_2(x^0), \\ &dh_m(x^0), dL_f h_m(x^0), \cdots, dL_f^{r_m-1} h_m(x^0). \end{aligned}$$

are linearly independent.

Proof Recall in the previous lecture, we introduced a Lemma:

$$\langle dL_f^s \phi(x), ad_f^{k+r} g(x) \rangle = \sum_{i=0}^{r} (-1)^i \binom{r}{i} L_f^{r-i} \langle dL_f^{s+i} \phi(x), ad_f^k g(x) \rangle, \quad \forall s, k, r \geq 0.$$

$\phi(x)$ defined on $U \subset R^n$

and as a consequence, the following are equivalent:

i) $L_g \phi(x) = L_g L_f \phi(x) = \cdots = L_g L_f^k \phi(x) = 0, \quad \forall x \in U$

ii) $L_g \phi(x) = L_{ad_f g} \phi(x) = \cdots = L_{ad_f^k g} \phi(x) = 0, \quad \forall x \in U.$

Therefore, $\langle dL_f^{k_1} h_i(x), ad_f^{k_2} g_j \rangle = \sum_{l=0}^{k_2} (-1)^l \binom{k_2}{l} L_f^{k_2-l} \underbrace{\langle dL_f^{l+k_1} h_i(x), ad_f^0 g(x) \rangle}_{= L_g L_f^{l+k_1} \phi(x)}$ → max k_1+k_2.

$= 0, \quad$ if $\quad k_1 + k_2 \leq r_i - 2, \quad \forall x \in U(x^0)$

$\langle dL_f^{k_1} h_i(x), ad_f^{k_2} g_j \rangle = (-1)^{r_i-1-k_1} L_g L_f^{r_i-1} h(x^0), \quad$ if $k_1 + k_2 = r_i - 1$

$$\Rightarrow \begin{bmatrix} dh_1(x) \\ \vdots \\ dh_m(x) \\ \vdots \\ dL_f^{r_1-1} h_1 \\ \vdots \\ dL_f^{r_m-1} h_m \end{bmatrix} \underbrace{[\overbrace{g_1(x) \cdots g_m(x)}^{m}, ad_f g_1, \cdots ad_f g_m, \cdots, ad_f^{r_1-1} g_1, \cdots, ad_f^{r_1-1} g_m]}_{n \times (r_1 \cdot m)}$$

(Without loss of generality, suppose $r_1 \geq r_2 \geq \cdots \geq r_m$)

$(-1)^{r_2-1} L_{g_1} L_f^{r_2-1} h_2 \quad (-1)^{r_2-1} L_{g_m} L_f^{r_2-1} h_2 \quad (-1)^{r_1-1} L_{g_1} L_f^{r_1-1} h_1(x^0) \quad (-1)^{r_1-1} L_{g_m} L_f^{r_1-1} h_1(x^0)$

$$= \left[\begin{array}{c|c|c|c} \begin{matrix} \overset{0}{\langle dh_1, g_1 \rangle} \cdots \overset{0}{\langle dh_1, g_m \rangle} \\ \overset{0}{\langle dh_2, g_1 \rangle} \cdots \\ \cdots \quad \cdots \end{matrix} & \cdots & \begin{matrix} 0 \quad\quad 0 \\ \underbrace{\langle dh_2, ad_f^{r_2-1} g_1 \rangle, \cdots \langle dh_2, ad_f^{r_2-1} g_m \rangle} \\ * \quad\quad * \end{matrix} & \begin{matrix} \underbrace{\langle dh_1, ad_f^{r_1-1} g_1 \rangle, \cdots \langle dh_1, ad_f^{r_1-1} g_m \rangle} \\ * \quad \cdots \quad * \\ * \quad\quad * \end{matrix} \\ \hline \vdots & \vdots & \vdots & \vdots \\ \hline \underbrace{\boxed{\begin{matrix} \langle dL_f^{r_1-1} h_1, g_1 \rangle \cdots \langle dL_f^{r_1-1} h_1, g_m \rangle \\ \vdots \quad\quad \vdots \\ \langle dL_f^{r_m-1} h_1, g_1 \rangle \cdots \langle dL_f^{r_m-1} h_m, g_m \rangle \end{matrix}}}_{A(x)} & & & \end{array}\right]$$

In other words, the matrix multiplication has a triangular structure whose diagonal blocks consist of the rows of nonsingular matrix $A(x)$.

Thus the matrix multiplication has full row rank, and therefore the statement is proved. ($\text{rank}(AB) \leq \min(\text{rank}(A), \text{rank}(B))$)

full row rank → full row rank

The above Lemma gives an <u>interesting fact about relative degree</u>: $\sum_{i=1}^{m} r_i \le n$

<u>Exact linearization via feedback</u>

Consider the affine nonlinear system $\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i$

Find $u_i = \alpha_i(x) + \sum_{j=1}^{m} \beta_{ij}(x) v_j$, $1 \le i, j \le m$, $\alpha_i(x), \beta_{ij}(x)$ are smooth functions defined on an open subset of $\mathbb{R}^n$.

$$\Rightarrow_{(*)} \dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)\left[\alpha_i(x) + \sum_{j=1}^{m} \beta_{ij}(x) v_j\right] = f(x) + \sum_{i=1}^{m} g_i(x)\alpha_i(x) + \sum_{i=1}^{m} g_i(x) \sum_{j=1}^{m} \beta_{ij}(x) v_j$$

$$= f(x) + g(x)\underbrace{[\alpha(x) + \underbrace{\beta(x)}_{\text{nonsingular}} v]}_{\text{"true" control happens.}}, \quad \alpha(x) = \begin{bmatrix} \alpha_1(x) \\ \vdots \\ \alpha_m(x) \end{bmatrix}, \ \beta(x) = \begin{bmatrix} \beta_{11}(x), & \cdots & \beta_{1m}(x) \\ \vdots & & \\ \beta_{m1}(x), & \cdots & \beta_{mm}(x) \end{bmatrix}, \ v = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix}$$

Problem: Find coordinate change $z = \Phi(x)$, such that $\dot{z} = Az + Bv$, where (A, B) is controllable

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<u>Lemma 2</u> Suppose the matrix $g(x^0)$ has rank m. Then, the system $(*)$ is exactly feedback linearizable iff there exists a neighbourhood $U(x^0)$ and m real-valued functions $h_1(x), \cdots, h_m(x)$, defined on $U(x^0)$, such that the system

$$\begin{aligned} \dot{x} &= f(x) + g(x) u \\ y &= h(x) \end{aligned}$$

has <u>relative degree $(r_1, \cdots, r_m)$ at x^0, and $\sum_{i=1}^{m} r_i = n$.</u>

<u>proof</u> (Sufficiency)

Suppose there exists such $h_1, \cdots, h_m$ to let the system has relative degree $(r_1, \cdots, r_m)$ and $\sum_{i=1}^{m} r_i = n$.

$$\Rightarrow \xi_k^i = \phi_k^i(x) = L_f^{k-1} h_i(x), \quad 1 \le k \le r_i, \quad 1 \le i \le m.$$

$$\Rightarrow \dot{\xi}_1^i = \frac{d}{dt}\left(L_f^0 h_i(x)\right) = \frac{d}{dt}(h_i(x)) = \frac{\partial h_i}{\partial x}\left(f(x) + \sum_{j=1}^{m} g_j(x) \cdot u_j\right) = L_f h_i(x) + \underbrace{\sum_{j=1}^{m} L_{g_j} h_i(x)}_{=0} \cdot u_j$$

$$\dot{\xi}_2^i = \frac{d}{dt}\left(L_f h_i(x)\right) = \frac{\partial L_f h_i(x)}{\partial x}\left(f(x) + g(x) \cdot u\right) = L_f^2 h_i(x) + \sum_{j=1}^{m} \underbrace{L_{g_j} L_f h_i(x)}_{=0} u_j$$

$$\vdots$$

$$\dot{\xi}_{r_i - 1}^i = \xi_{r_i}$$

$$\dot{\xi}_{r_i} = \frac{d}{dt}\left(L_f^{r_i - 1} h_i(x)\right) = \frac{\partial L_f^{r_i-1} h_i(x)}{\partial x}\left(f(x) + g(x) u\right) = b_i(\xi) + \sum_{j=1}^{m} \frac{\partial L_f^{r_i-1} h_i(x)}{\partial x} g_j(x) u_j$$

$$= b_i(\xi) + \sum_{j=1}^{m} L_{g_j} L_f^{r_i - 1} h_i(x) \cdot u_j = b_i(\xi) + \underbrace{[A(x)]_i}_{\text{the } i\text{'th row of } A(\xi)} u$$

in $U(x^0)$

$\Rightarrow$ Since $A(x)$ is nonsingular (<u>recall the definition of relative degree</u>)

we can choose $u = A(x)^{-1}[-b(\xi) + v]$ to cancel the "unwanted" $b(\xi)$ term

The system becomes

$$\begin{aligned} \dot{\xi}_1^i &= \xi_2^i \\ &\vdots \\ \dot{\xi}_{r_i-1} &= \xi_{r_i}^i \\ \dot{\xi}_{r_i}^i &= v_i \end{aligned}$$

$\forall \ 1 \le i \le m$, which is clearly controllable.

(Necessity) We first show the relative degree $(r_1, \cdots r_m)$ remain unchanged under feedback

Recall that in the previous lecture, we showed that $L^k_{f+g\alpha} h_i(x) = L^k_f h_i(x)$, $0 \le k \le r_i - 1$, $1 \le i \le m$

From this, we conclude that

$$\underbrace{L_{(g\beta)_j}}_{} \underbrace{L^k_{f+g\alpha} h_i(x)}_{=0} = L_{(g\beta)_j} L^k_f h_i(x) = \frac{\partial L^k_f h_i(x)}{\partial x} (g\beta)_j = \frac{\partial L^k_f h_i(x)}{\partial x} \sum_{s=1}^m g_s \beta_{sj}(x)$$

$$= \sum_{s=1}^m L_{g_s} L^k_f h_i(x) \beta_{sj}(x) = 0, \quad \forall\ 0 \le k \le r_i - 2, \ 1 \le i,j \le m, \ x \in U(x^0)$$

j th column $= \sum_{s=1}^m g_s \beta_{sj}$

$$[g_1 \cdots g_m] \begin{bmatrix} \beta_{11} & \cdots & \beta_{1m} \\ \vdots & & \vdots \\ \beta_{m1} & \cdots & \beta_{mm} \end{bmatrix}$$

$$\Rightarrow [L_{(g\beta)_1} \underbrace{L^{r_i-1}_{f+g\alpha} h_i(x^0)}_{= L_f^{r_i-1} h_i}, \ldots, L_{(g\beta)_m} L^{r_i-1}_{f+g\alpha} h_i(x^0)]$$

$$= \underbrace{[L_{g_1} L_f^{r_i-1} h_i(x^0), \ldots, L_{g_m} L_f^{r_i-1} h_i(x^0)]}_{i\text{th row of } A(x)} \cdot \beta(x^0)$$

Hence if $\beta(x^0)$ is nonsingular, $[L_{(g\beta)_1} L^{r_i-1}_{f+g\alpha} h_i(x^0), \ldots, L_{(g\beta)_m} L^{r_i-1}_{f+g\alpha} h_i(x^0)] \neq 0$. $\neq 0$

$\Rightarrow$ <u>The relative degree $(r_1, \cdots, r_m)$ remain unchanged under feedback!</u>

We may assume that A, B are in the form of

$$A = \mathrm{diag}(A_1, \cdots, A_m), \quad B = \mathrm{diag}(b_1, \cdots, b_m)$$

where $\underbrace{A_i}_{K_i \times K_i} = \begin{bmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ 0 & \cdots & \cdots & 0 \end{bmatrix}$ $\underbrace{b_i}_{K_i \times 1} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$, since we can always to a linear coordinate change $(\bar{z} = Tz)$ and feedback to attain it.

We can decompose $z = \phi(x)$ as $\underbrace{z}_{n \times 1} = \begin{bmatrix} z^1 \\ \vdots \\ z^m \end{bmatrix} \begin{matrix} \}K^1 \\ \\ \}K^m \end{matrix}$ and $K_1 + K_2 + \cdots K_m = n$

and set $y_i = (1 \ 0 \cdots 0) z^i$.

Such system $\dot{z} = Az + Bv$ would have relative degree $(K_1, \cdots K_m)$, where $\sum_{i=1}^m K_i = n$

Recall in the previous lecture, we introduced an intepretation of finding function $\lambda(x)$ such that it has relative degree n. That is solving a set of differential equations of the form $L_g L^k_f \lambda_i(x) = 0$, $0 \le k \le n-2$, $L_g L^{n-1}_f \lambda_i(x^0) \neq 0$, $x \in U(x^0)$

<u>We shall extend this result to the MIMO case.</u>

Recall the following useful Lemma we covered in the previous lecture:

<u>Lemma 3</u>. Let ϕ be a real-valued function and f, g vector fields, all defined in $U \subseteq \mathbb{R}^n$. Then for any choice of $s, k, r \ge 0$, it holds

$$\langle dL^s_f \phi(x), ad^{k+r}_f g(x) \rangle = \sum_{i=0}^r (-1)^i \binom{r}{i} L^{r-i}_f \langle dL^{s+i}_f \phi(x), ad^k_f g(x) \rangle.$$

And consequently, the following are equivalent:

i) $L_g \phi(x) = L_g L_f \phi(x) = \cdots = L_g L^k_f \phi(x) = 0, \ \forall x \in U$

ii) $L_g \phi(x) = L_{ad_f g} \phi(x) = \cdots = L_{ad^k_f g} \phi(x) = 0, \ \forall x \in U$

For MIMO case, the condition regarding exact feedback linearization would be regarding the distributions spanned by vector fields of the form

$$g_1 \cdots g_m,\ ad_f\, g_1, \cdots, ad_f\, g_m, \cdots, ad_f^{n-1} g_1, \cdots, ad_f^{n-1} g_m .$$

Denote the distributions

$G_0 = \text{span}\{g_1, \cdots, g_m\}$

$G_1 = \text{span}\{g_1, \cdots, g_m, ad_f\, g_1, \cdots, ad_f\, g_m\}$

$\vdots$

$G_i = \text{span}\{ad_f^k g_j : 0 \le k \le i,\ 1 \le j \le m\}, \quad \forall\, i = 0, 1, \cdots, n-1.$

☆ Thm Suppose the matrix $g(x^0)$ has rank m. Then, the nonlinear affine system is exactly feedback linearizable iff (also called "nonsingular")

(i) the distribution G_i has constant dimension near x^0, $\forall\, 0 \le i \le n-1$.

(ii) the distribution G_{n-1} has dimension n.

(iii) the distribution G_i is involutive, $\forall\, 0 \le i \le n-2$.

proof. (Sufficiency)

The main issue is to find $\lambda_1(x), \cdots \lambda_m(x)$ such that $L_{g_j} L_f^k \lambda_i(x) = 0$, $\forall\, 0 \le k \le r_i - 2$, $1 \le j \le m$,

and $A(x) = \begin{bmatrix} L_{g_1} L_f^{r_1-1} h_1(x^0), & \cdots, & L_{g_m} L_f^{r_1-1} h_m(x^0) \\ \vdots & & \vdots \\ L_{g_1} L_f^{r_m-1} h_1(x^0), & \cdots & L_{g_m} L_f^{r_m-1} h_m(x^0) \end{bmatrix}$ is nonsingular.

Using Lemma 3, $L_{g_j} L_f^k \lambda_i(x) = 0$, $0 \le k \le r_i - 2$, $1 \le j \le m$, $x \in U(x^0)$

$\Leftrightarrow L_{ad_f^k g_j} \lambda_i(x) = \langle d\lambda_i(x), ad_f^k g_j(x) \rangle = 0$, $0 \le k \le r_i - 2$, $1 \le j \le m$, $x \in U(x^0)$

$\Rightarrow d\lambda_i(x)$ must be a covector belonging to the co-distribution

$$G_{r_i-2}^{\perp} = (\text{span}\{ad_f^k g_j : 0 \le k \le r_i - 2,\ 1 \le j \le m\})^{\perp}$$

By (i), $G_0, \cdots G_{n-1}$ all have constant dimension near x^0, and by (ii) $\dim(G_{n-1}) = n$.

$\Rightarrow \exists\, K \le n$, s.t. $\dim(G_{K-2}) < n$, $\dim(G_{K-1}) = n$

$G_0, G_1, \cdots \underset{\dim < n}{G_{K-2}}, \underset{\dim = n}{G_{K-1}}, \cdots, \underset{\dim = n}{G_{n-1}}$

Denote $m_1 = n - \dim(G_{K-2})$

By (iii), G_{K-2} is involutive, therefore by Frobenius theorem, there exist m_1 functions $\{\lambda_i(x)\}$, $i = 1, \cdots, m_1$, such that $\text{span}\{d\lambda_i : 1 \le i \le m_1\} = G_{K-2}^{\perp}$.

Namely, these functions satisfy $\langle d\lambda_i(x), ad_f^k g_j(x) \rangle = 0$, $\forall\, x \in U(x^0)$, $0 \le k \le K-2$, $1 \le j \le m$, $1 \le i \le m_1$

$G_{K-2} = \{g_1, \cdots g_m, ad_f\, g_1, \cdots ad_f\, g_m, \cdots ad_f^{K-2} g_1, \cdots ad_f^{K-2} g_m\}$

By Lemma 3, it is equivalent to $\underline{L_{g_j} L_f^k \lambda_i(x) = 0}^{(**)}$, $\forall x \in U(x^0)$, $0 \le k \le \kappa-2$, $1 \le j \le m$, $1 \le i \le m_1$.
This gives the fact that the $m_1 \times m$ matrix $A^1(x) = \{a_{ij}^1(x)\} = \{L_{g_j} L_f^{\kappa-1} \lambda_i(x)\}$ has rank m_1 at x^0.

To see that, suppose this is not the case (contradiction proof).
Then, using $(**)$ and again Lemma 3, we have that

$$\sum_{i=1}^{m_1} c_i L_{g_j} L_f^{\kappa-1} \lambda_i(x^0) = \sum_{i=1}^{m_1} (-1)^{\kappa-1} c_i \langle d\lambda_i(x^0), ad_f^{\kappa-1} g_j(x^0) \rangle = 0$$

$$(s=0,\ r=\kappa-1,\ k=0)$$

$$\langle d\lambda, ad_f^{\kappa-1} g \rangle = \sum_{i=0}^{\kappa-1} (-1)^i \binom{\kappa-1}{i} L_f^{\kappa-1-i} \langle dL_f^i \lambda, g \rangle = (-1)^{\kappa-1} L_g L_f^{\kappa-1} \lambda$$

$$(\langle dL_f^i \lambda, g \rangle = 0,\ \forall 0 \le i \le \kappa-2)$$

$\forall\ 1 \le j \le m$, for some real numbers c_i, $1 \le i \le m_1$.
But this, together with $\langle d\lambda_i, ad_f^k g_j \rangle = 0$, $\forall x \in U(x^0)$, $1 \le j \le m$, $1 \le i \le m_1$ (k varies)
implies that

$$\sum_{i=1}^{m_1} c_i \langle d\lambda_i(x^0), ad_f^k g_j(x^0) \rangle = 0, \quad \forall\ 0 \le k \le \kappa-1, \quad 1 \le j \le m.$$

This shows that $\sum_{i=1}^{m_1} c_i d\lambda_i(x^0) \in G_{\kappa-1}^{\perp}(x^0)$ $\rightsquigarrow$ $\dim(G_{\kappa-1}) = n \Rightarrow$ the vector must be 0,
$\Rightarrow c_1 = c_2 = \cdots = c_{m_1} = 0$ (since $d\lambda_i$ is linearly independent)

As a summary, $\underset{m_1 \times m}{A^1(x^0)} = \{L_{g_j} L_f^{\kappa-1} \lambda_i(x^0)\}$ has full row rank.

Note that $\underline{m_1 \le m}$. (since $A^1(x^0)$ is $m_1 \times m$, and has full row rank)

① If $m_1 = m$, then these functions $\lambda_i(x)$ indeed solves the problem.
Because $(**) \Rightarrow$

$$A^1(x^0) = \begin{bmatrix} L_{g_1} L_f^{\kappa-1} \lambda_1(x^0), & \cdots, & L_{g_m} L_f^{\kappa-1} \lambda_m(x^0) \\ \vdots & & \\ L_{g_1} L_f^{\kappa-1} \underset{\lambda_m(x^0)}{\lambda_{m_1}(x^0)}, & \cdots & L_{g_m} L_f^{\kappa-1} \underset{\lambda_m(x^0)}{\lambda_{m_1}(x^0)} \end{bmatrix} = A(x^0),$$

with $r_1 = r_2 = \cdots = r_m = \kappa$.
Thus the system with outputs $\lambda_i(x)$, $1 \le i \le m$, has relative degree $(\kappa, \cdots \kappa)$.
Moreover, by the fact that the sum of relative degrees should be smaller than n, namely, $m\kappa \le n$, and $n = \dim(G_{\kappa-1}) \le m\kappa$. $\left(G_{\kappa-1} = \mathrm{span}\{g_1, \cdots g_m, ad_f g_1, \cdots ad_f g_m, ad_f^{\kappa-1} g_1, \cdots ad_f^{\kappa-1} g_m\}\right)$
$\Rightarrow m\kappa = n$, The $\lambda_i(x)$ (by construction) would let the system have relative degree $(\kappa, \cdots \kappa)$, and $m\kappa = n$. $\Rightarrow$ exact feedback linearizable

② If $m_1 < m$, $\{\lambda_i(x), i = 1, \cdots, m_1\}$ only provides a part of the solution,
We have to continue searching for additional $m - m_1$ new functions.
Idea move a step backward and look at $G_{\kappa-3}$, try to find new functions among those differentials that spans $G_{\kappa-3}^{\perp}$.

Before we proceed, we would like to show

a) the codistribution $\Omega_1 = \text{span}\{d\lambda_1, \cdots d\lambda_{m_1}, dL_f\lambda_1, \dots, dL_f\lambda_{m_1}\}$ has dimension $2m_1$ around x^0.

b) $\Omega_1 \subset G_{\kappa-3}^{\perp}$.

Since $G_{\kappa-3} = \{g_1, \cdots g_m, ad_f g_1, \cdots ad_f g_m, \cdots ad_f^{\kappa-3} g_1, \cdots ad_f^{\kappa-3} g_m\}$

$G_{\kappa-2} = \{g_1, \cdots g_m, ad_f g_1, \cdots ad_f g_m, \cdots, ad_f^{\kappa-2} g_1, \cdots ad_f^{\kappa-2} g_m\}$, $\Rightarrow G_{\kappa-3} \subset G_{\kappa-2}$

$\Rightarrow G_{\kappa-2}^{\perp} \subset G_{\kappa-3}^{\perp} \Rightarrow d\lambda_i \in G_{\kappa-3}^{\perp}, \; i=1,\cdots,m_1$

On the otherhand, Recall $(**)$, it holds that $L_{g_j} L_f^k \lambda_i(x) = 0, \; \forall x \in U(x^0), \; 0 \le k \le \kappa-2, \; 1 \le j \le m, \; 1 \le i \le m_1$

and by Lemma 3, $\langle dL_f\lambda_i, ad_f^k g_j \rangle = \sum_{l=0}^{k} (-1)^l \binom{k}{l} L_f^{k-l} \langle dL_f^{l+1}\lambda_i(x), g \rangle$

$= \sum_{l=0}^{k} (-1)^l \binom{k}{l} L_f^{k-l} \underbrace{L_g L_f^{l+1} \lambda_i(x)}_{=0}$

hence for $0 \le k \le \kappa-3$, $1 \le j \le m$, $1 \le i \le m_1$, we have

$\langle dL_f\lambda_i(x), ad_f^k g_j(x) \rangle = 0 \quad \Rightarrow dL_f\lambda_i(x) \in G_{\kappa-3}^{\perp}, \; i=1,\cdots,m_1$

$\Rightarrow \Omega_1 \subset G_{\kappa-3}^{\perp} \Rightarrow$ b) proved.

To prove a), suppose this is not the case, then there exists numbers c_i, d_i; $1 \le i \le m_1$, s.t. $\sum_{i=1}^{m_1} (c_i d\lambda_i(x^0) + d_i dL_f\lambda_i(x^0)) = 0$

$\Rightarrow \langle \sum_{i=1}^{m_1} (c_i d\lambda_i(x^0) + d_i dL_f\lambda_i(x^0)), ad_f^{\kappa-2} g_j(x^0) \rangle = 0, \; j=1,\cdots,m.$

$\Rightarrow \sum_{i=1}^{m_1} c_i \underbrace{\langle d\lambda_i(x^0), ad_f^{\kappa-2} g_j(x^0) \rangle}_{=0, \text{ since } d\lambda_i \in G_{\kappa-2}^{\perp}} + d_i \underbrace{\langle dL_f\lambda_i(x^0), ad_f^{\kappa-2} g_j(x^0) \rangle}_{\text{Lemma 3}} = 0$

Lemma 3, $s=0$, $k+r=\kappa-2$:

$\langle dL_f^s\phi(x), ad_f^{k+r+1} g(x) \rangle = L_f \langle dL_f^s\phi(x), ad_f^{k+r} g(x) \rangle - \langle dL_f^{s+1}\phi(x), ad_f^{k+r} g(x) \rangle$

$\Rightarrow \sum_{i=1}^{m_1} d_i \left[L_f \underbrace{\langle d\lambda_i(x^0), ad_f^{\kappa-2} g_j(x^0) \rangle}_{=0} - \langle d\lambda_i(x^0), ad_f^{\kappa-1} g_j(x^0) \rangle \right]$

$\Rightarrow \sum_{i=1}^{m_1} d_i \langle d\lambda_i(x^0), ad_f^{\kappa-1} g_j(x^0) \rangle = 0$

Recall the proof for $A^1(x)$ is full row rank, this gives $d_i = 0, \; i=1,\dots,m_1$

$\left.\begin{array}{l} \Rightarrow \sum_{i=1}^{m_1} c_i d\lambda_i(x^0) = 0 \\ d\lambda_i \text{ is linearly independent} \end{array}\right\} \Rightarrow c_i = 0, \; i=1,\cdots,m_1,$

$\Rightarrow$ a) is proved.

From a) & b), we know $\dim(G_{\kappa-3}^{\perp}) \ge 2m_1$. Suppose now it is strictly larger,

Set $m_2 = \dim(G_{\kappa-3}^{\perp}) - 2m_1$, namely, $m_2 > 0$

Since by Assumption iii), $G_{\kappa-3}$ is involutive, by Frobenius theorem, $G_{\kappa-3}^{\perp}$ is spanned by $2m_1 + m_2$ exact one-forms.

a) and b) already characterize $2m_1$ such exact one-forms. (those that spans Ω_1)

Thus we can conclude that there exist m_2 additional functions, $\lambda_i(X)$, $m_1+1 \le i \le m_1+m_2$,

such that $\underbrace{G_{K-3}^{\perp} = \underset{\dim 2m_1}{\Omega_1} + \text{span}\{d\lambda_i(x),\ m_1+1 \le i \le m_1+m_2\}}$. $\rightsquigarrow \dim m_2$

Note that, these new functions $\lambda_i(X)$, $m_1+1 \le i \le m_1+m_2$, are such that

$$L_{g_j} L_f^k \lambda_i(x) = 0,\ \forall x \in U(x^0),\quad 0 \le k \le K-3,\quad 1 \le j \le m,\ m_1+1 \le i \le m_1+m_2.$$

$$\rightsquigarrow L_{ad_f^k g_j} \lambda_i = 0 \Leftrightarrow \underbrace{\langle d\lambda_i, ad_f^k g_j \rangle = 0}_{0 \le k \le K-3} \quad \text{since } d\lambda_i \in G_{K-3}^{\perp}.$$

Now we claim that

c) the $(m_1+m_2)\times m$ matrix

$$\mathcal{A}^2(x) = \begin{array}{c} \left[\begin{array}{ccc} \langle d\lambda_1(x), ad_f^{K-1} g_1(x)\rangle, & \cdots, & \langle d\lambda_1(x), ad_f^{K-1} g_m(x)\rangle \\ \vdots & & \\ \langle d\lambda_{m_1}, ad_f^{K-1} g_1(x)\rangle, & \cdots & \langle d\lambda_{m_1}(x), ad_f^{K-1} g_m(x)\rangle \\ \hline \langle d\lambda_{m_1+1}(x), ad_f^{K-2} g_1(x)\rangle, & \cdots, & \langle d\lambda_{m_1+1}(x), ad_f^{K-2} g_m(x)\rangle \\ \vdots & & \\ \langle d\lambda_{m_1+m_2}(x), ad_f^{K-2} g_1(x)\rangle, & \cdots & \langle d\lambda_{m_1+m_2}(x), ad_f^{K-2} g_m(x)\rangle \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{array}{c}a\\a\\a\end{array}}\right\} m_1 \\ \left.\vphantom{\begin{array}{c}a\\a\\a\end{array}}\right\} m_2 \end{array} \\ \underbrace{\hphantom{\langle d\lambda_{m_1+m_2}(x), ad_f^{K-2} g_1(x)\rangle, \cdots \langle d\lambda_{m_1+m_2}(x), ad_f^{K-2} g_m(x)\rangle}}_{m} \end{array}$$

has rank equal to m_1+m_2 at x^0.

To prove this, suppose there exists real numbers c_i, $1 \le i \le m_1$, d_i, $m_1+1 \le i \le m_1+m_2$, such that

$$-\sum_{i=1}^{m_1} c_i \underline{\langle d\lambda_i(x^0), ad_f^{K-1} g_j(x^0)\rangle} + \sum_{i=m_1+1}^{m_1+m_2} d_i \langle d\lambda_i(x^0), ad_f^{K-2} g_j(x^0)\rangle = 0$$

using Lemma 3) again, we have

$$\left\langle \sum_{i=1}^{m_1} c_i\, dL_f \lambda_i(x^0) + \sum_{i=m_1+1}^{m_1+m_2} d_i\, d\lambda_i(x^0),\ ad_f^{K-2} g_j(x^0)\right\rangle = 0$$

$$\Rightarrow \sum_{i=1}^{m_1} c_i\, dL_f \lambda_i(x^0) + \sum_{i=m_1+1}^{m_1+m_2} d_i\, d\lambda_i(x^0) \in \underbrace{(\text{span}\{ad_f^{K-2} g_j(x^0) : 1 \le j \le m\})}_{\subset G_{K-2}}{}^{\perp}$$

$$\Rightarrow \sum_{i=1}^{m_1} c_i\, dL_f \lambda_i(x^0) + \sum_{m_1+1}^{m_1+m_2} d_i\, d\lambda_i(x^0) \in G_{K-2}^{\perp}$$

$$\rightsquigarrow = \text{span}\{d\lambda_1, \cdots, d\lambda_{m_1}\}.$$

$\Rightarrow$ contradiction, $G_{K-3}^{\perp} = \underset{\substack{\| \\ \text{span}\{d\lambda_1, \cdots, d\lambda_{m_1}, dL_f\lambda_1, \cdots dL_f\lambda_{m_1}\}}}{\Omega_1} + \text{span}\{d\lambda_i(x),\ m_1+1 \le i \le m_1+m_2\}$

$\dim(\Omega_1) = 2m_1$

$\Rightarrow d\lambda_1, \cdots d\lambda_{m_1}, dL_f\lambda_1, \cdots dL_f\lambda_{m_1}$ are all linearly independent.

$\Rightarrow dL_f\lambda_1, \cdots, dL_f\lambda_{m_1}$ and $\text{span}\{d\lambda_i(x), m_1+1 \le i \le m_1+m_2\}$ can not spanned by $\{d\lambda_i(x), i = 1, \cdots, m_1\}$ unless $c_i = 0$, $\underline{d_i = 0}$

$\Rightarrow c_i = 0$, $d_i = 0$, $\forall i$ $\Rightarrow$ c) is proved.

Note that $m_1+m_2 \le m$, since $\mathcal{A}^2(x^0)$ has full row rank,

If $m_1+m_2 = m$, we can infer that the system has relative degree $\{r_1, \cdots r_m\}$, with $r_1 = \cdots = r_{m_1} = K$.

$r_{m_1+1} = \cdots = r_m = K-1$,

Moreover, $r_1+r_2+\cdots+r_m = n$, since

$$n = \dim(\underset{\substack{\downarrow \\ \text{span}\{g_1, \cdots, g_m, ad_f g_1, \cdots, ad_f g_m, \cdots, ad_f^{K-2} g_1, \cdots ad_f^{K-2} g_m\}}}{G_{K-2}}) + m_1 \le \underset{\rightsquigarrow m_1+m_2}{m}(K-1) + m_1 = \underbrace{m_1 K + m_2(K-1)}_{\rightsquigarrow \text{sum of relative degree is less than } n} \le n.$$

If m_1+m_2 is strictly less than m, (this includes the case of $m_2=0$), one has to continue seaching for additional functions that spans $G_{k-4}^{\perp}$.

After $k-1$ iterations of this, one has found m_{k-1} functions

$$\begin{cases} d\lambda_i(x), dL_f\lambda_i(x), \cdots, dL_f^{k-2}\lambda_i(x), & \text{for } 1\le i\le m_1 \\ d\lambda_i(x), dL_f\lambda_i(x), \cdots, dL_f^{k-3}\lambda_i(x), & \text{for } m_1+1\le i\le m_1+m_2 \\ \cdots & \\ d\lambda_i(x), dL_f\lambda_i(x), & \text{for } m_1+\cdots+m_{k-3}+1\le i\le m_1+\cdots+m_{k-2} \\ d\lambda_i(x) & \text{for } m_1+\cdots+m_{k-2}+1\le i\le m_1+\cdots+m_{k-1}. \end{cases}$$

They are basis of $G_0^{\perp}$. Recall that $G_0=\{g_1, \cdots g_m\}$, it has dimension m by assumption,

$$\Rightarrow \underline{n-m = \dim(G_0^{\perp}) = (k-1)m_1+(k-2)m_2+\cdots+m_{k-1}}.$$

$G_{k-1}^{\perp}(x^0)=\mathrm{span}\{d\lambda_i,\ i=1,\cdots,m_1\}$.

$\Omega_1=\mathrm{span}\{d\lambda_1,\cdots d\lambda_{m_1}, \underbrace{dL_f\lambda_1,\cdots dL_f\lambda_{m_1}}_{\text{one order higher}}\}$

$\dim(\Omega_1)=2m_1$

"Add Lie derivative, and they are linearly independent".

We can do the same, it is possible to prove the following vectors

$$\begin{cases} d\lambda_i(x), dL_f\lambda_i(x), \cdots, dL_f^{k-2}\lambda_i(x), dL_f^{k-1}\lambda_i(x), & 1\le i\le m_1 \\ d\lambda_i(x), dL_f\lambda_i(x), \cdots, dL_f^{k-3}\lambda_i(x), dL_f^{k-2}\lambda_i(x), & m_1+1\le i\le m_1+m_2 \\ \cdots & \\ d\lambda_i(x), dL_f\lambda_i(x), dL_f^2\lambda_i(x), & m_1+\cdots+m_{k-3}+1\le i\le m_1+\cdots+m_{k-2} \\ d\lambda_i(x), dL_f\lambda_i(x), & m_1+\cdots+m_{k-2}+1\le i\le m_1+\cdots+m_{k-1} \end{cases}$$

are linearly independent in $U(x^0)$.

$$\Rightarrow n-(km_1+(k-1)m_2+\cdots+2m_{k-1})\geq 0.$$

If the inequality strictly holds, let $m_k = n-(km_1+(k-1)m_2+\cdots+2m_{k-1})$

$$\begin{aligned} m_1+m_2+\cdots+m_k &= m_1+m_2+\cdots+m_{k-1}+n-(km_1+(k-1)m_2+\cdots+2m_{k-1}) \\ &= n-[(k-1)m_1+(k-2)m_2+\cdots+m_{k-1}] \end{aligned}$$

$$\Rightarrow m_1+m_2+\cdots+m_k = m.$$

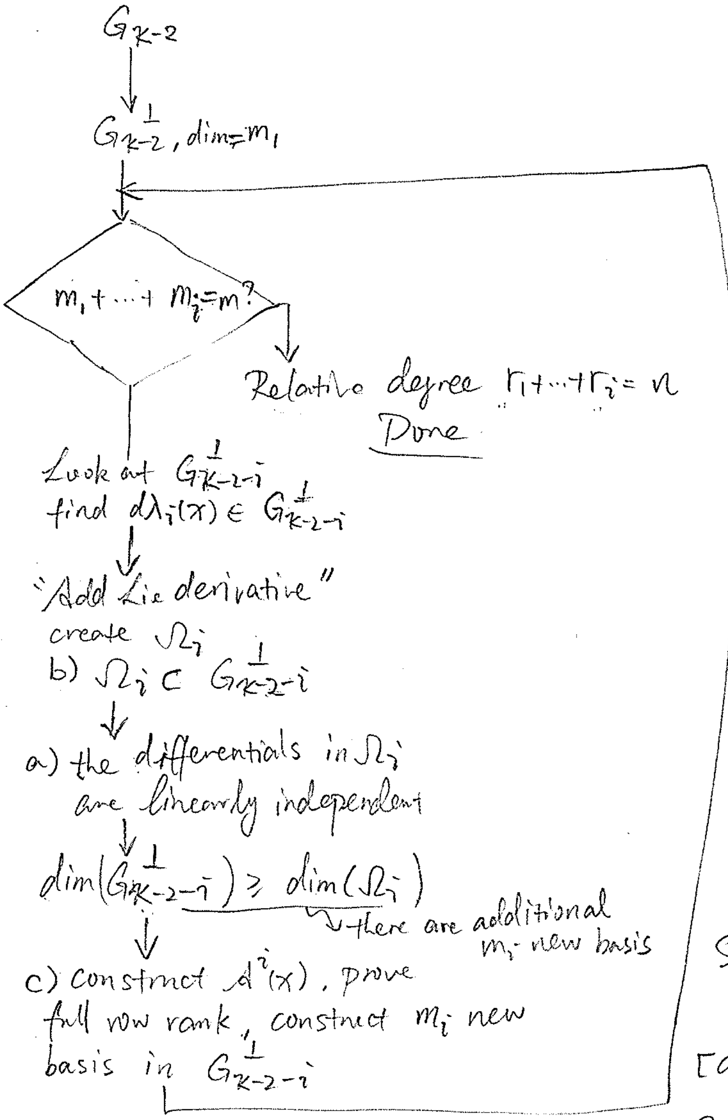
$\Rightarrow$ there exists m_k functions $\lambda_i(x)$, $m_1+\cdots+m_{k-1}+1\le i\le m$ such that they together with those in the table form exactly n independent differentials in $U(x^0)$.

Using arguments similar to c), it is possible to prove the system, with outputs $\lambda_i(x)$, $1\le i\le m$ has relative degree $(r_1,\cdots,r_m)$ at x^0, with

$$\begin{cases} r_i = k, & \text{for } 1\le i\le m_1 \\ r_i = k-1, & \text{for } m_1+1\le i\le m_1+m_2 \\ \cdots & \\ r_i = 2, & \text{for } m_1+\cdots+m_{k-2}+1\le i\le m_1+\cdots+m_{k-1} \\ r_i = 1, & \text{for } m_1+\cdots+m_{k-1}+1\le i\le m. \end{cases}$$

And $r_1+r_2+\cdots+r_m=n$. proof for sufficiency complete.

The proof for necessity is omitted here.



<u>Ex</u> $\dot{x} = \begin{bmatrix} x_2 + x_2^2 \\ x_3 - x_1x_4 + x_4x_5 \\ x_2x_4 + x_1x_5 - x_5^2 \\ x_5 \\ x_2^2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cos(x_1 - x_5) \\ 0 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} u_2.$

In this system, $G_0 = \text{span}\{g_1, g_2\}$, $\dim(G_0) = 2 = m$, in a neighbourhood of $x^0 = 0$.

Since $[g_1, g_2] = 0 \Rightarrow G_0$ is involutive.

$G_1 = \text{span}\{g_1, g_2, ad_f g_1, ad_f g_2\}$

$ad_f g_1(x) = \begin{bmatrix} 0 \\ -\cos(x_1 - x_5) \\ -x_2 \sin(x_1 - x_5) \\ 0 \\ 0 \end{bmatrix}$, $ad_f g_2(x) = \begin{bmatrix} 0 \\ -1 \\ -(x_1 - x_5) \\ -1 \\ 0 \end{bmatrix}$

$\dim(G_1(x^0)) = 4$, nonsingular.

Since $[g_1, ad_f g_1] = [g_1, ad_f g_2] = [g_2, ad_f g_1] = [g_2, ad_f g_2] = 0$

$[ad_f g_1, ad_f g_2] = \tan(x_1 - x_5) g_1(x) \Rightarrow G_1$ involutive

$G_2 = \text{span}\{g_1, g_2, ad_f g_1, ad_f g_2, ad_f^2 g_1, ad_f^2 g_2\}$

$\dim(G_2(x^0)) = 5$,

Since $G_{i-1} \subset G_i$, $\dim(G_2) = 5 = n \Rightarrow G_2 = G_3 = G_4$, G_2 and G_3 are involutive.

$k = 3$ (Since $G_2 = n$), we have to first consider $G_1^{\perp}$. $\dim(G_1^{\perp}) = 1$

There exists $\lambda_1(x)$ s.t $\text{span}\{d\lambda_1\} = G_1^{\perp}$

choose $\lambda_1(x) = x_1 - x_5$, "Add Lie derivative"

$\text{span}\{d\lambda_1(x), dL_f\lambda_1(x)\} \subset G_0^{\perp}$

$d\lambda_1(x) = (1\ 0\ 0\ 0\ -1)$, $dL_f\lambda_1(x) = dx_2 = (0\ 1\ 0\ 0\ 0)$.

Choose $\lambda_2(x)$ whose differential is linearly independent of $d\lambda_1(x)$ and $dL_f\lambda_1(x)$ and is annihilated by the vectors of G_0, $\lambda_2(x) = x_4$ is a good choice.

It is easy to check $L_{g_1}\lambda_1(x) = L_{g_2}\lambda_1(x) = L_{g_1}L_f\lambda_1(x) = L_{g_2}L_f\lambda_1(x) = 0$

$L_{g_1}\lambda_2(x) = L_{g_2}\lambda_2(x) = 0$

$\begin{bmatrix} L_{g_1}L_f^2\lambda_1(x) & L_{g_2}L_f^2\lambda_1(x) \\ L_{g_1}L_f\lambda_2(x) & L_{g_1}L_f\lambda_2(x) \end{bmatrix}$ is nonsingular at $x = 0$.

$\Rightarrow$ the system with $y_1 = \lambda_1(x)$, $y_2 = \lambda_2(x)$ will have relative degree $(r_1, r_2) = (3, 2)$

$r_1 + r_2 = 5 = n$.