Nonlinear Control Theory

Lecture 13. Feedback stabilization II

Last time

- Feedback stabilization.
  - zero dynamics
  - Feedback stabilizability
  - Local asymptotic stabilization
  - Idea of extending to MIMO
  - Global vs local.

Today

- Passivity approach
- Arstein-Sontag's theorem.
- Back stepping.

Consider the system

$$(*)\quad \begin{aligned} \dot{x} &= f(x) + g(x)u, \\ y &= h(x), \end{aligned} \qquad \begin{aligned} &x \in N(0) \subseteq \mathbb{R}^n,\ f(0)=0,\ f\in C^1,\ g\in C^1. \\ &h(0)=0,\ \underline{y\in\mathbb{R}^m,\ u\in\mathbb{R}^m} \end{aligned}$$

"square system"

Def. The system $(*)$ is said to be passive if there exists a positive semidefinite function $V(x)$, (also called storage function), such that for all $u\in\mathcal{U}$, it holds that $y^T u \geq \frac{\partial V}{\partial x}(f+gu)$.

"inflow power"

$\boxed{\begin{aligned} &V(0)=0 \\ &V(X)\geq 0 \text{ in } \mathcal{D}\setminus\{0\} \end{aligned}}$

It is said to be lossless if $y^T u = \frac{\partial V}{\partial x}(f+gu)$.

It is said to be strictly passive if $y^T u \geq \frac{\partial V}{\partial x}(f+gu) + S(x)$, where $S(x)$ is positive definite.

Def The system is zero-state observable if no solution of $\dot{x}=f(x)$ can stay identically in the set $\{x \mid h(x)=0\}$ except the trivial solution $x(t)\equiv 0$.

Thm If the system $(*)$ is passive with a radially unbounded positive definite storage function $V(x)$, and is zero-state observable, then the origin $x=0$ can be globally stabilized by $u=-\phi(y)$, where ϕ is any locally Lipschitz function such that $\phi(0)=0$ and $y^T\phi(y)>0\ \forall y\neq 0$.

proof Use the storage function $V(x)$ as a Lyapunov function

$$\dot{V}(x) = \frac{\partial V}{\partial x}\left(f(x) - g(x)\phi(y)\right) \leq -y^T\phi(y) \leq 0, \quad \text{thus } \dot{V}(x)<0 \text{ if } y\neq 0$$

$\dot{V}(x)=0$ iff $y=0$. By zero-state observability,

$$y(t)\equiv 0 \Rightarrow \phi(y)\equiv 0 \Rightarrow u=\phi(y)=0 \Rightarrow x(t)\equiv 0.$$

$\Rightarrow$ By Lasalle's invariance principle, the statement is proved

Thm. If the system (*) is strictly passive with a radially unbounded positive definite storage function, then the system is globally stabilizable.

We continue to consider the nonlinear affine system $\dot{x} = f(x) + g(x)u$ (**) $\boxed{u \in \mathbb{R}}$

Def A positive definite, radially unbounded and differentiable function $V(x)$ is called a control Lyapunov function (CLF) if $\forall x \neq 0$, it holds

$$L_g V(x) = 0 \Rightarrow L_f V(x) < 0$$

can be extended to multi-input case

Intepretation

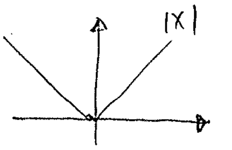
Recall that, by converse Lyapunov theorem, if there exists a global stabilizing controller $u = \alpha(x)$, then there exists a positive definite, radially unbounded differentiable function $V(x)$, such that $\frac{\partial V(x)}{\partial x}(f(x) + g(x)\alpha(x)) = L_f V(x) + L_g V(x) \alpha(x) < 0$ for each $x \neq 0$.

For those $L_g V(x) = 0$, $L_f V(x)$ must be strictly negative.

The above intepretation shows that the existence of a CLF is necessary for the existence of a global stabilizing controller.

Q: To what extent this is also sufficient?

A: It turns out if we are satisfied with "almost smooth" feedback controllers, this condition is also sufficient.



Def. (Almost smooth functions)

A function $\alpha(x)$ defined on $\mathbb{R}^n$ is called almost smooth if $\alpha(0) = 0$, α is smooth on $\mathbb{R}^n \setminus \{0\}$ and at least continuous at $x = 0$.

smooth $\rightarrow C^\infty$.

Thm. Consider the system (**), in which $f(x)$ and $g(x)$ are smooth vector fields and $f(0) = 0$. There exists an almost smooth feedback law $u = \alpha(x)$ which globally asymptotically stabilizes the equilibrium $x = 0$ iff there exists a positive definite and radially unbounded smooth function $V(x)$ with the following properties:

i) $L_g V(x) = 0$ implies $L_f V(x) < 0 \quad \forall x \neq 0$

ii) for each $\varepsilon > 0$, there exists $\delta > 0$ such that, if $x \neq 0$ satisfies $\|x\| < \delta$, then there is some u with $|u| < \varepsilon$ such that $L_f V(x) + L_g V(x) \cdot u < 0$

proof. (Necessity) The necessary condition i) is derived, as shown above from the converse Lyapunov theorem.

The necessary condition ii) is a simple consequence of the hypothesis that the stabilizing feedback control $u = \alpha(x)$ is continuous at $x=0$ (the $u=\alpha(x)$ is the u in ii)).

(Sufficiency)

To prove the sufficiency, consider the following open subset of $\mathbb{R}^2$.

$$S = \{(a,b) \in \mathbb{R}^2 \mid b > 0, \text{ or } a < 0\}$$

and define on S a function $\phi(a,b)$ as follows:

$$\phi(a,b) = \begin{cases} 0, & \text{if } b=0 \text{ and } a<0 \\ \dfrac{a+\sqrt{a^2+b^2}}{b} & \text{otherwise.} \end{cases}$$

Set $F(a,b,p) = bp^2 - 2ap - b$ and note that $F(a,b,p)=0$ is satisfied by $p = \phi(a,b)$ $\forall (a,b) \in S$.

And the Jacobian $\left[\dfrac{\partial F}{\partial p}\right]_{p=\phi(a,b)} = 2(b\phi(a,b)-a) \neq 0 \quad \forall (a,b) \in S$

Hence by implicit function theorem, the solution $p = \phi(a,b)$ is real-analytic.

Now suppose $V(x)$ is a function satisfies i).

Let $a = L_f V(x)$, $b = [L_g V(x)]^2$, and $(a,b) \in S$.

Let $\alpha(x) = \begin{cases} 0, & \text{if } x=0 \\ -L_g V(x)\,\phi(L_f V(x), [L_g V(x)]^2) & \text{otherwise.} \end{cases}$

$\alpha(x)$ is a composition of the real-analytic function $\phi(\cdot,\cdot)$ and smooth functions $L_f V(x)$, $L_g V(x)$, it is indeed smooth on $\mathbb{R}^n \setminus \{0\}$.

With ii), it is possible to show that $\alpha(x)$ is continuous at $x=0$ (how?)

Thus $\alpha(x)$ is almost smooth.

It holds that $\dfrac{\partial V}{\partial x}(f(x)+g(x)\alpha(x)) = L_f V(x) - L_g V(x) \cdot \dfrac{L_f V(x) + \sqrt{(L_f V(x))^2 + (L_g V(x))^4}}{L_g V(x)}$

$$= -\sqrt{[L_f V(x)]^2 + [L_g V(x)]^4} < 0, \quad \forall x \neq 0.$$

Thus, this feedback law globally asymptotically stabilizes $x=0$ of the system.

Back stepping

Ex | Integrator backstepping. Consider a special normal form:

$$\dot{\eta} = f(\eta) + g(\eta)\xi, \qquad \eta \in \mathbb{R}^n, \xi \in \mathbb{R}, u \in \mathbb{R}.$$
$$\dot{\xi} = u, \qquad f, g \in C^\infty \text{ (Smooth functions)}, f(0)=0$$

First consider partially the system $\dot{\eta} = f(\eta) + g(\eta)\xi$, and view ξ as the control input of it.

Suppose the system can be stabilized by $\xi = \phi(\eta)$, with $\phi(0) = 0$, with a known corresponding Lyapunov function $V_1(\eta)$ such that

$$\dot{V}_1(\eta) = \frac{\partial V_1}{\partial \eta}[f(\eta) + g(\eta)\phi(\eta)] \leq -W(\eta), \text{ where } W(\eta) \text{ is a positive definite function.}$$

Hence by adding and subtract, we can transform the original system as

$$\dot{\eta} = [f(\eta) + g(\eta)\phi(\eta)] + g(\eta)[\xi - \phi(\eta)]$$
$$\dot{\xi} = u.$$

Use change of variable $z = \xi - \phi(\eta)$, we have

$$\dot{\eta} = [f(\eta) + g(\eta)\phi(\eta)] + g(\eta)\cdot z$$
$$\dot{z} = \underbrace{u - \dot{\phi}(\eta)}_{:= V}$$

"back stepping" using integrator

Note $\dot{\phi} = \frac{\partial \phi}{\partial \eta}[f(\eta) + g(\eta)\xi]$.

Construct Lyapunov function $V_2(\eta, \xi) = V_1(\eta) + \frac{1}{2}z^2$

$$\Rightarrow \dot{V}_2(\eta,\xi) = \frac{\partial V_1}{\partial \eta}[f(\eta) + g(\eta)\phi(\eta) + g(\eta)z] + z\cdot V$$
$$= \frac{\partial V_1}{\partial \eta}[f(\eta) + g(\eta)\phi(\eta)] + \frac{\partial V_1}{\partial \eta}g(\eta)z + zV \leq -W(\eta) + \frac{\partial V_1}{\partial \eta}g(\eta)z + zV$$

Choose $V = -\frac{\partial V_1}{\partial \eta}g(\eta) - kz$, $z = \xi - \phi(\eta)$, where $k > 0$, yields

$$\dot{V}_2(\eta,\xi) \leq -W(\eta) - kz^2 \Rightarrow \eta = 0,\ z = 0 \text{ is asymptotically stable.}$$

$$\left.\begin{array}{l} z = \xi - \phi(\eta) = 0 \\ \phi(0) = 0,\ \eta = 0 \end{array}\right\} \Rightarrow \xi = 0,\ \eta = 0 \text{ is asymptotically stable.}$$

Ex]
$$\dot{x}_1 = x_1^2 - x_1^3 + x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = u$$

Backstepping x_3 is seen as the control input of

$$\dot{x}_1 = x_1^2 - x_1^3 + x_2$$
$$\dot{x}_2 = x_3.$$

This can be globally stabilized by $x_3 = \underbrace{-x_1 - (1+2x_1)(x_1^2 - x_1^3 + x_2) - (x_2 + x_1 + x_1^2)}_{\phi(x_1, x_2)}$

and $V_1(x_1, x_2) = \frac{1}{2}x_1^2 + \frac{1}{2}(x_2 + x_1 + x_1^2)^2$

let $z_3 = x_3 - \phi(x_1, x_2)$

$$\Rightarrow \dot{x}_1 = x_1^2 - x_1^3 + x_2$$
$$\dot{x}_2 = \phi(x_1, x_2) + z_3$$
$$\dot{z}_3 = u - \frac{\partial \phi}{\partial x_1}(x_1^2 - x_1^3 + x_2) - \frac{\partial \phi}{\partial x_2}(\phi + z_3)$$

Construct $V_2(x_1, x_2, z) = V_1(x_1, x_2) + \frac{1}{2}z_3^2$, we have

$$\dot{V}_2(x_1, x_2, z) = \frac{\partial V_1}{\partial x_1}(x_1^2 - x_1^3 + x_2) + \frac{\partial V_1}{\partial x_2}(z_3 + \phi) + z_3\left(u - \frac{\partial \phi}{\partial x_1}(x_1^2 - x_1^3 + x_2) - \frac{\partial \phi}{\partial x_2}(\phi + z_3)\right)$$

$$= \underbrace{-x_1^2 - x_1^4 - (x_2 + x_1 + x_1^2)^2}_{\leq 0} + z_3\left[\frac{\partial V_1}{\partial x_2} - \frac{\partial \phi}{\partial x_1}(x_1^2 - x_1^3 + x_2) - \frac{\partial \phi}{\partial x_2}(z_3 + \phi) + u\right]$$

Taking $u = \underbrace{-\frac{\partial V_1}{\partial x_2} + \frac{\partial \phi}{\partial x_1}(x_1^2 - x_1^3 + x_2) + \frac{\partial \phi}{\partial x_2}(z_3 + \phi)}_{\text{these things cancels the "unwanted terms"}} - z_3$

$\Rightarrow \dot{V}_2(x_1, x_2, z) = -x_1^2 - x_1^4 - (x_2 + x_1 + x_1^2) - z_3^2 < 0.$

$\Rightarrow x_1 = 0,\ x_2 = 0,\ z_3 = 0$ is globally asymptotically stable.

$\Rightarrow z_3 = 0 = x_3 - \underbrace{\phi(\overset{0}{x_1}, \overset{0}{x_2})}_{=0} \Rightarrow x_3 = 0. \Rightarrow \begin{matrix} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{matrix}$ is globally asymptotically stable.

Now consider the more general form.

$\dot{\eta} = f(\eta) + g(\eta)\xi$

$\dot{\xi} = f_a(\eta, \xi) + g_a(\eta, \xi)u$, where f_a and g_a are smooth (C^∞), $u \in \mathbb{R}$.

The normal form is also affine

"$g_a(\eta, \xi) \neq 0$" (definition of relative degree)

We can choose $u = \frac{1}{g_a(\eta, \xi)}[u_a - f_a(\eta, \xi)]$ to tranform the system into

$\begin{cases} \dot{\eta} = f(\eta) + g(\eta)\xi \\ \dot{\xi} = u_a \end{cases}$ that are now in the same form as above.

Namely, if we choose $u = \frac{1}{g_a(\eta, \xi)}\left[\frac{\partial \phi}{\partial \eta}[f(\eta) + g(\eta)\xi] - \frac{\partial V_1}{\partial \eta}g(\eta) - k[\xi - \phi(\eta)] - f_a(\eta, \xi)\right]$

$(u_a = \dot{\phi} + v,\ v = -\frac{\partial V_1}{\partial \eta}g(\eta) - kz,\ z = \xi - \phi(\eta))$

and choose $V_2(\eta, \xi) = V_1(\eta) + \frac{1}{2}[\xi - \phi(\eta)]^2$ as a stabilizing feedback control and Lyapunov function.

Extending this idea, we can recursively apply this backstepping technique, and stabilize the system of the form:

$$
\begin{aligned}
\dot{x} &= f_0(x) + g_0(x)z_1 \\
\dot{z}_1 &= f_1(x, z_1) + g_1(x, z_1)z_2 \\
\dot{z}_2 &= f_2(x, z_1, z_2) + g_2(x, z_1, z_2)z_3 \\
&\ \vdots \\
\dot{z}_{k-1} &= f_{k-1}(x, z_1, \cdots, z_{k-1}) + g_{k-1}(x, z_1, \cdots z_{k-1})z_k \\
\dot{z}_k &= f_k(x, z_1, \cdots, z_k) + g_k(x, z_1, \cdots, z_k)u
\end{aligned}
$$

, $x \in \mathbb{R}^n$, $z_1, \cdots, z_k \in \mathbb{R}$

$f_0, \cdots f_k$ all vanish at origin

Assume $g_i(x, z_1, \cdots, z_i) \neq 0$, $1 \le i \le k$.

① First consider $\dot{x} = f_0(x) + g_0(x)z_1$. Assume we have found a control $z_1 = \phi_0(x)$ with $\phi_0(0) = 0$ that stabilizes the system and $V_0(x)$ such that

$\frac{\partial V_0}{\partial x}[f_0(x) + g_0(x)\phi_0(x)] \le -W(x)$ positive definite function.

② With $\phi_0(x)$ and $V_0(x)$, consider

$$\begin{cases} \dot{x} = f_0(x) + g_0(x) z_1 \\ \dot{z}_1 = f_1(x, z_1) + g_1(x, z_1) z_2 \end{cases}$$

with $\eta = x$, $\xi = x_1$, $u = z_2$, $f = f_0$, $g = g_0$, $f_a = f_1$, $g_a = g_1$

$k_1 > 0$

Use the result in the above example, we choose $z_2 = \phi_1(x_1, z_1) = \frac{1}{g_1}\left[\frac{\partial \phi_0}{\partial x}(f_0 + g_0 z_1) - \frac{\partial V_0}{\partial x} g_0 - k_1(z_1 - \phi) - f_1\right]$

and choose $V_1(x, z_1) = V_0(x) + \frac{1}{2}(z_1 - \phi_0(x))^2$

③ Next, consider

$$\dot{x} = f_0(x) + g_0(x) z_1$$
$$\dot{z}_1 = f_1(x, z_1) + g_1(x, z_1) z_2$$
$$\dot{z}_2 = f_2(x, z_1, z_2) + g_2(x, z_1, z_2) z_3$$

$\eta = \begin{bmatrix} x \\ z_1 \end{bmatrix}$, $\xi = z_2$, $u = z_3$, $f = \begin{bmatrix} f_0 + g_0 z_1 \\ f_1 \end{bmatrix}$, $g = \begin{bmatrix} 0 \\ g_1 \end{bmatrix}$, $f_a = f_2$, $g_a = g_2$

$k_2 > 0$

Use the result in the above example,

choose $z_3 = \phi_2(x, z_1, z_2) = \frac{1}{g_2}\left[\frac{\partial \phi_1}{\partial x}(f + g_0 z_1) + \frac{\partial \phi_1}{\partial z_1}(f_1 + g_1 z_2) - \frac{\partial V_1}{\partial z_1} g_1 - k_2(z_1 - \phi_1) - f_2\right]$

and $V_2(x, z_1, z_2) = V_1(x, z_1) + \frac{1}{2}(z_2 - \phi_2(x, z_1))^2$

Repeat this k times until we get $u = \phi_k(x, z_1, \cdots, z_k)$ and the Lyapunov function $V_k(x, z_1, \cdots, z_k)$

Ex) Suppose the normal form has the "special form"

$$\dot{\eta} = q(\xi, \eta) \overset{\Delta}{=} f_0(\eta) + g_0(\eta) z_1$$
$$\dot{z}_1 = z_2$$
$$\vdots$$
$$\dot{z}_{r-1} = z_r$$
$$\dot{z}_r = b(\xi, \eta) + a(\xi, \eta) u$$

has exactly the form in the above example.

Robust stabilization.

Consider

$$\dot{\eta} = f(\eta) + g(\eta)\xi + \delta_\eta(\eta, \xi)$$
$$\dot{\xi} = f_a(\eta, \xi) + g_a(\eta, \xi) u + \delta_\xi(\eta, \xi),$$

$\eta \in \mathbb{R}^n$, $\xi \in \mathbb{R}$, $g_a(\eta, \xi) \neq 0$. All functions are smooth. δ_η, δ_ξ be uncertain terms.

We assume f and f_a vanish at the origin, and it holds for the uncertain terms that

$$\left.\begin{array}{l} \|\delta_\eta(\eta, \xi)\|_2 \le a_1 \|\eta\|_2 \\ |\delta_\xi(\eta, \xi)| \le a_2 \|\eta\|_2 + a_3 |\xi| \end{array}\right\}$$

restricts the class of uncertainties.

Starting with $\dot{\eta} = f(\eta) + g(\eta)\xi + \delta_\eta(\eta, \xi)$, suppose we can find a stabilizing feedback control $\xi = \phi(\eta)$ with $\phi(0) = 0$ and a smooth, positive definite Lyapunov function $V_1(\eta)$, such that

$$\frac{\partial V_1}{\partial \eta}[f(\eta) + g(\eta)\phi(\eta) + \underbrace{\delta_\eta(\eta, \xi)}_{\le a_1 \|\eta\|^2}] \le -b\|\eta\|_2^2$$

$\Rightarrow$ $\eta = 0$ is asymptotically stable equilibrium of $\dot{\eta} = f(\eta) + g(\eta)\phi(\eta) + \delta_\eta(\eta, \xi)$.

Suppose further $\phi(\eta)$ satisfies:

$$|\phi(\eta)| \le a_4 \|\eta\|_2, \quad \left\|\frac{\partial \phi}{\partial \eta}\right\|_2 \le a_5.$$

Now construct $V_2(\eta,\xi) = V_1(\eta) + \frac{1}{2}[\xi - \phi(\eta)]^2$

$$\dot{V}_2 = \frac{\partial V_1}{\partial \eta}[f(\eta) + g(\eta)\xi + \delta_\eta(\eta,\xi)] + (\xi - \phi(\eta))[f_a(\eta,\xi) + g_a(\eta,\xi)u + \delta_\xi(\eta,\xi) - \frac{\partial \phi}{\partial \eta}(f(\eta) + g(\eta)\xi + \delta_\eta(\eta,\xi))]$$

$$= \frac{\partial V_1}{\partial \eta}[f + g\phi + \delta_\eta] + \frac{\partial V_1}{\partial \eta} g(\xi - \phi) + (\xi - \phi)[f_a + g_a u + \delta_\xi - \frac{\partial \phi}{\partial \eta}(f + g\xi + \delta_\eta)]$$

Taking $u = \frac{1}{g_a}[\frac{\partial \phi}{\partial \eta}(f + g\xi) - \frac{\partial V_1}{\partial \eta} g - f_a - k(\xi - \phi)]$

$$\Rightarrow \dot{V}_2 = \underbrace{\frac{\partial V_1}{\partial \eta}[f + g\phi + \delta_\eta]}_{\le -b\|\eta\|^2} + \cancel{\frac{\partial V_1}{\partial \eta} g(\xi - \phi)} + (\xi - \phi)[\cancel{f_a} + \frac{\partial \phi}{\partial \eta}\cancel{(f + g\xi)} - \cancel{\frac{\partial V_1}{\partial \eta} g} - \cancel{f_a} - k(\xi - \phi) - \frac{\partial \phi}{\partial \eta}\cancel{(f + g\xi)} - \frac{\partial \phi}{\partial \eta}\delta_\eta + \delta_\xi]$$

$$\le -b\|\eta\|_2^2 + (\xi - \phi)[\delta_\xi - \frac{\partial \phi}{\partial \eta}\delta_\eta] - k(\xi - \phi)^2 = -b\|\eta\|^2 + (\xi - \phi)\delta_\xi - (\xi - \phi)\frac{\partial \phi}{\partial \eta}\delta_\eta - k(\xi - \phi$$

$$\le -b\|\eta\|_2^2 + |\xi - \phi|\underbrace{|\delta_\xi|}_{\le a_2\|\eta\|_2 + a_3|\xi|} + |\xi - \phi|\underbrace{|\frac{\partial \phi}{\partial \eta}|}_{\le a_5}\cdot\underbrace{|\delta_\eta|}_{\le a_1\|\eta\|_2} - k(\xi - \phi)^2$$

Cauchy-Swartz

$$\le -b\|\eta\|^2 + |\xi - \phi|(a_2\|\eta\|_2 + a_3|\xi|) + a_1 a_5|\xi - \phi|\|\eta\|_2 - k(\xi - \phi)^2$$

$$= -b\|\eta\|^2 + a_2\|\eta\|_2|\xi - \phi| + a_3|\xi - \phi + \phi|\cdot|\xi - \phi| + a_1 a_5|\xi - \phi|\,\|\eta\|_2 - k(\xi - \phi)^2$$

Triangular inequality

$$\le -b\|\eta\|^2 + a_2\|\eta\|_2|\xi - \phi| + a_3(\xi - \phi)^2 + a_3\underbrace{|\phi|}_{\le a_4\|\eta\|_2}\cdot|\xi - \phi| + a_1 a_5|\xi - \phi|\,\|\eta\|_2 - k(\xi - \phi)^2$$

$$\le -b\|\eta\|^2 + \underbrace{(a_2 + a_4 + a_1 a_5)}_{:= a_6 > 0}\|\eta\|_2|\xi - \phi| - (k - a_3)(\xi - \phi)^2$$

$$= -\begin{bmatrix}\|\eta\|_2 \\ |\xi - \phi|\end{bmatrix}^T \begin{bmatrix} b & -a_6 \\ a_6 & k - a_3\end{bmatrix}\begin{bmatrix}\|\eta\| \\ |\xi - \phi|\end{bmatrix} \quad \text{choose } k > a_3 + \frac{a_6^2}{b}$$

This would yield $\dot{V}_2 \le -\sigma[\|\eta\|_2^2 + |\xi - \phi|^2]$ for some $\sigma > 0$.