Nonlinear Control Theory

Lecture 4. Lyapunov Direct Method. I.

Last time

- Concepts of stability
- Analysis via linearization
- Lyapunov function

Today

- General Lyapunov function method (time-varying systems)
- Stability theorems.

For autonomous systems, $\dot{x} = f(x)$, (asymptotic) stability $\Rightarrow$ uniform (asymptotic) stability

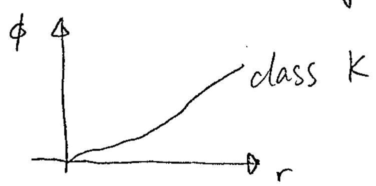
Find an $V(x) \geq 0$ and $V(x) > 0 \ \forall x \in \mathcal{D} \setminus \{0\}$, $\dot{V}(x) \leq 0 \Rightarrow$ stability

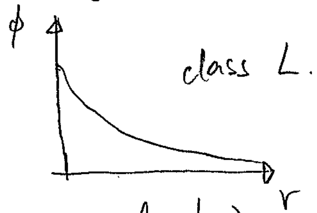
$\dot{V}(x) < 0 \Rightarrow$ asymptotic stability

For time-varying systems, more delicate Lyapunov theorems are needed.

Def A function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is of class K if it is continuous, strictly increasing and $\phi(0) = 0$;

it is of class L if it is continuous, strictly decreasing, $\phi(0) < \infty$, and $\lim_{r \to \infty} \phi(r) = 0$.





Def (Various function classes)

- A continuous function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is said to be a locally positive definite function (lpdf), if: $V(t, 0) = 0 \ \forall t \geq 0$ and there exists a $r > 0$ and $\alpha \in K$ s.t. $\alpha(\|x\|) \leq V(t, x)$, $\forall t \geq 0$, $\forall x \in B_r$.
- Continuous V is decrescent if there exists a function β of class K, s.t. $V(t, x) \leq \beta(\|x\|)$, $\forall t \geq 0$, $\forall x \in B_r$
- Continuous V is positive definite if $r = \infty$
- V is radially unbounded if $V(t, x) \geq \varphi(\|x\|)$, $\lim_{r \to \infty} \varphi(r) = \infty$

Consider a continuous function $W: \mathbb{R}^n \mapsto \mathbb{R}$, and $V(t,x) \geq W(x)$, then $V(t,x)$ is a (locally) positive definite function if $W(x)$ is (locally) positive definite.

(Naturally, we assume $V(t,0) = 0$)

How do we decide lpdf for a time-invariant function?

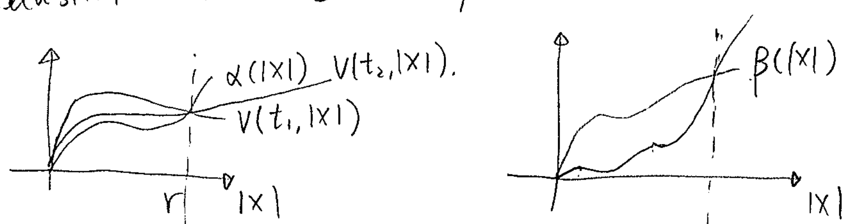
- $W(0) = 0$
- $W(x) > 0 \quad \forall x \in B_r \setminus \{0\}$

An lpdf $W(x)$ is also decrescent. (why?)

For pdf, it should also satisfy:

- $\exists c > 0$, s.t. $\inf\limits_{|x|>c} W(x) > 0$

Illustrate with 1D examples



Locally positive definite. Decrescent

Ex 1.) $V(x) = x_1^2 + x_2^2$ is positive definite. in $\mathbb{R}^2$

2) $V(t,x) = x_1^2(1+\sin^2 t) + x_2^2(1+\cos^2 t)$ is positive definite and decrescent.

3) $V(t,x) = (t+1)(x_1^2 + x_2^2)$ is positive definite, but not decrescent.

4). $V(x) = x_1^2 + \sin^2 x_2$ is lpdf, but not pdf.

Consider $\dot{x} = f(t,x)$, $x \in \mathbb{R}^n$, $f: [0,\infty) \times \mathcal{D} \mapsto \mathbb{R}^n$ is piecewise continuous in t and locally Lipschitz in x on $[0,\infty) \times \mathcal{D}$, $\mathcal{D}$ contains $x=0$.

The origin $x=0$ is an equilibrium at $t=0$ if $f(t,0) = 0$, $\forall t \geq 0$.

Total derivative $\quad \dot{V}(t,x) := \frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x} f(x,t)$

Thm (Critical stability) $x=0$ is uniformly stable if there exists a continuously differentiable (C^1) decresent lpdf $V(t,x)$ such that

$$\dot{V} \leq 0 \quad \forall t \geq 0 \text{ in a neighbourhood of } 0.$$

Ex. $\dot{x}_1 = a(t) x_2^{2n+1}$

$\dot{x}_2 = -a(t) x_1^{2n+1}$

Let $V(x) = \frac{1}{2}(x_1^{2n+2} + x_2^{2n+2})$. $\dot{V}(x) = (n+1) x_1^{2n+1} \cdot \dot{x}_1 + (n+1) x_2^{2n+1} \cdot \dot{x}_2$

lpdf.

$= (n+1) \cdot x_1^{2n+1} \cdot a(t) x_2^{2n+1} + (n+1) x_2^{2n+1} (-a(t) x_1^{2n+1})$

$= 0$.

Thm (Instability)

$x=0$ is unstable if there exists a C^1, decrescent function $V(t,x)$ and $t_0 \geq 0$, such that

1). $\dot{V}$ is lpdf

2) $V(t,0) = 0 \quad \forall t \geq t_0$

3) there exists a sequence $\{x_n \neq 0\}$ where $x_n \to 0$ as $n \to \infty$ such that $V(t_0, x_n) \geq 0$

In particular, if $V(t,x)$ is lpdf, it automatically satisfies 2) & 3).

Ex. $\dot{x} = x^n$, $x \in \mathbb{R}$

1). n is odd. $V(x) = \frac{1}{2}x^2 \Rightarrow \dot{V}(x) = x^{n+1}$ done, not stable.

2) n is even. $V(x) = x. \Rightarrow \dot{V}(x) = x^n$ lpdf.

construct sequence $x_n = \frac{1}{n}$, $V(x_n) = \frac{1}{n} > 0$. not stable.

Thm (Asymptotic Stability)

$x=0$ is uniformly asymptotically stable if there exists a decrescent lpdf $V(t,x)$ s.t $-\dot{V}$ is lpdf. $\{\alpha(\|x\|) \leq V(t,x) \leq \beta(\|x\|)$.

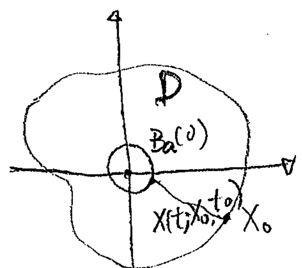
$-\dot{V}(t,x) \geq \gamma(\|x\|)$. $\}$.

Domain of attraction

$D(0) = \{x_0 \in \mathbb{R}^n ; \lim_{t \to \infty} x(t; x_0, t_0) = 0\}$.

Fact: the domain of attraction is always an open set.

$\lim_{t \to \infty} x(t; x_0, t_0) = 0 \Leftrightarrow \forall a > 0, \exists T > 0$, s.t. $x(t; x_0, t_0) \in B_a(0) \quad \forall t > T$.



$\Rightarrow x(t; x_0, t_0)$ is a continuous mapping regarding x_0.

The preimage of an open set regarding a continuous mapping is also open.

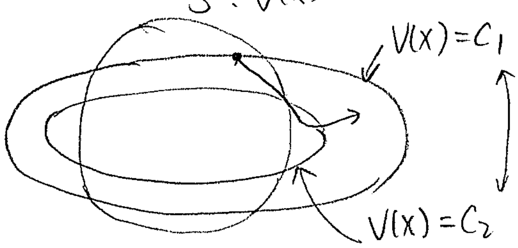
what happens on the $\partial\mathcal{D}(0)$? Only contains trajectories, but does not exclude finite-escape time.

if a trajectory start on $\partial\mathcal{D}(0)$, it remains on $\partial\mathcal{D}(0)$.

Q : Suppose on a domain S, $V(X) > 0$, $\dot{V}(X) \le 0$, $\forall x \neq 0$ in S.

Does this imply that $S \subseteq \mathcal{D}(0)$ and/or S is invariant ?

No!



The level set of $V(X)$ is not aligned with S!

"Yes" if $\{x \mid V(x) \le c\}$ is bounded and contained in S.

Ex
$$\dot{x}_1 = -x_1 - x_2 + x_1^3 + x_1 x_2^2$$
$$\dot{x}_2 = -x_2 + 2x_1 + x_2^3 + x_1^2 x_2$$
$$V(x_1, x_2) = \tfrac{1}{2}(2x_1^2 + x_2^2). \Rightarrow \dot{V}(x) = (2x_1^2 + x_2^2)(x_1^2 + x_2^2 - 1) \le 0 \text{ for } \|X\| < 1$$
$$\left(\text{Try } X(0) = \begin{bmatrix} 0.7 \\ 0.7 \end{bmatrix}\right)$$

Thm (Exponential Stability)

Suppose there exists a lpdf $V(t, X)$ which is bounded by
$$a\|X\|^2 \le V(t, X) \le b\|X\|^2, \ \forall t \ge 0, \ \forall x \in B_r.$$

If $\dot{V}(t, x) \le -c\|X\|^2, \ \forall t \ge 0, \ \forall x \in B_r, \quad (a, b, c > 0)$

then $x = 0$ is exponentially stable.

proof Since $-b\|X\|^2 \le -V(t, X)$, we have
$$\dot{V}(t, X) \le -c\|X\|^2 \le -\frac{c}{b} V(t, X)$$
$$\Rightarrow V(t, X(t)) \le V(t_0, X_0) \cdot e^{-\frac{c}{b}(t - t_0)}$$

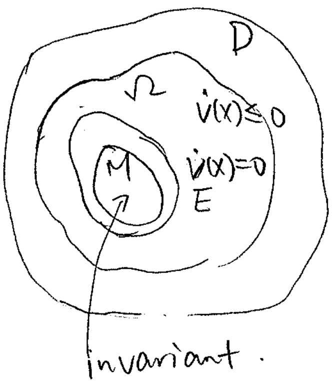
Since $\|X(t)\|^2 \le \frac{1}{a} V(t, X) \le \frac{1}{a} V(t_0, X_0) e^{-\frac{c}{b}(t - t_0)} \le \frac{b}{a} \|X_0\|^2 e^{-\frac{c}{b}(t - t_0)}$

$$\Rightarrow \|X(t)\| \le \sqrt{\frac{b}{a}} \|X_0\| \cdot e^{-\frac{c}{2b}(t - t_0)}$$

For autonomous system $\dot{x} = f(x)$, $(*)$ we have further results.

Thm (Lasalle's invariance principle)

Let $\Omega \subset D$ be a compact set (close & bounded) that is positively invariant with respect to $(*)$. Let $V: D \mapsto \mathbb{R}$ be C^1 function s.t $\dot{V}(x) \le 0$ in Ω, Let E be the set of all pts in Ω where $\dot{V}(x) = 0$, Let M be the largest invariant set in E, Then every solution starting in Ω approaches M as $t \to \infty$.



positively invariant : if a trajectory starts there, it would stay there as time goes.

Ex Consider

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 - f(x_2),$$

where $f(0) = 0$ and $x f(x) > 0 \quad \forall x \neq 0$.

Let $V = \frac{1}{2}(x_1^2 + x_2^2)$ radially unbounded.

$$\dot{V}(x) = x_1 \cdot x_2 + x_2 \cdot (-x_1 - f(x_2)) = -x_2 f(x_2) \le 0.$$

this holds $\forall x$

$\Rightarrow$ The level set $\{x \mid V(x) \le c\}$ is invariant $\forall c > 0$.

$\Rightarrow$ converge to the largest invariant set in $\{x \mid x_2 f(x_2) = 0\}$
$= \{x \mid x_2 = 0\}$

$$\left.\begin{aligned} \dot{x}_1 &= 0 \\ \dot{x}_2 &= -x_1 \end{aligned}\right\} \Rightarrow \text{The only possible invariant solution is } \begin{cases} x_1 \equiv 0 \\ x_2 \equiv 0 \end{cases}$$

$\Rightarrow$ $x = 0$ is globally asymptotically stable.