Nonlinear Control Theory.

Lecture 6. Stability of Invariant set. & Model Reduction.

Last time:

- Proof of Lasalle's invariance principle.
- The Lure's problem
- Global stability
- Converse Lyapunov theorems

Today

- Stability of invariant set
- Center manifold theory
- Singular perturbation.

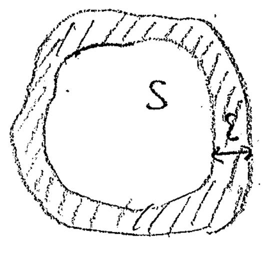
Consider $\dot{x} = f(x)$, $x \in \mathbb{R}^n$, f is Lipschitz continuous.

We define the distance for a point x_0 to a set S as:

$$\text{dist}(x_0, S) = \inf_{y \in S} \|x_0 - y\|.$$

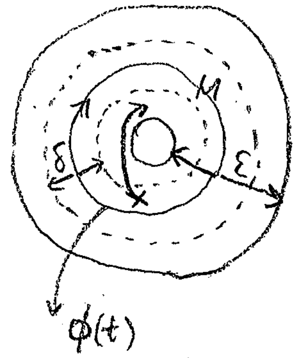
Now denote the ε-neighbourhood of S as:

$$U_\varepsilon = \{x \in \mathbb{R}^n \mid \text{dist}(x, S) < \varepsilon\}.$$



Def An invariant set M is stable if for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$x(0) \in U_\delta \Rightarrow x(t) \in U_\varepsilon$$



M is asymptotically stable if it is stable and there exists $\delta > 0$, such that

$$x(0) \in U_\delta \Rightarrow \lim_{t \to \infty} \text{dist}(x(t), M) = 0$$

Suppose $\phi(t)$ is a periodic solution of the system. Then the invariant set $\gamma = \{\phi(t) \mid 0 \le t \le T\}$ is called an orbit.

Def A (nontrivial) periodic solution $\phi(t)$ is orbitally stable if the orbit γ is stable; it is asymptotically orbitally stable if γ is asymptotically stable.

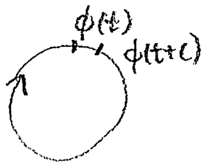
Is it possible to study the stability of the periodic solution by considering the stability of the "error dynamics"?

Let $e(t) = x(t) - \phi(t)$. $\Rightarrow$

$$\begin{aligned} \dot{e}(t) &= \dot{x}(t) - \dot{\phi}(t) = f(x(t)) - f(\phi(t)) \\ &= f(e(t) + \phi(t)) - f(\phi(t)). \\ &:= F(t, e(t)) \end{aligned}$$

Fact: $e = 0$ is never asymptotically stable.

choose c small enough, s.t $\phi(t+c) \in$ domain of attraction

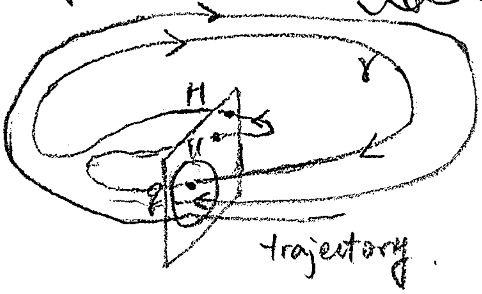


$$\begin{aligned} \dot{e}(t) &= \dot{\phi}(t+c) - \dot{\phi}(t) = f(\phi(t+c)) - f(\phi(t)) \\ &= f(\phi(t) + e(t)) - f(\phi(t)) \end{aligned}$$

$e(t) = \phi(t+c) - \phi(t)$ is periodic, i.e. $e(t+T) = e(t)$

Poincaré map

Let H be a hypersurface transversal to γ at a point p.
Let U be a neighbourhood of p on H, small enough such that γ intersects U only once at p



The first return or Poincaré map $P: U \to H$ is defined for $q \in H$ by:

$$P(q) = x(T; q)$$

where $T(q)$ is the time taken for the flow $x(T; q)$ starting from q, and return to H.

In particular, it holds

- $P(p) = p$. (recall the point p is on the orbit and U is a neighbourhood of p).
- $P(U)$ is a neighbourhood of p. $P: U \to P(U)$ is a diffeomorphism

equilibrium (fixed-pt)

Intuitively, the stability of p for the discrete-time system (or map P) reflects the stability of γ.

Formally, we have the following theorem.

Thm A periodic orbit γ is asymptotically stable if the corresponding discrete-time system obtained from Poincaré mapping is asymptotically stable.

In particular, if $\left.\frac{\partial P(x)}{\partial x}\right|_{x=p}$ has $n-1$ eigenvalues of modulus less than 1, then γ is asymptotically stable.

However, it is in general dificult to compute the Poincaré mapping. We can instead compute the eigenvalues of the linearized Poincaré mapping by linearizing the dynamical system around the limit cycle $\phi(t)$, namely,

$$\dot{x} = f(x) \xrightarrow{\text{linearization}} \dot{z} = \underbrace{\left.\frac{\partial f(x)}{\partial x}\right|_{x=\phi(t)}}_{A(t)} z$$

$A(t) \longrightarrow$ periodic with period T

Thm (Floquet)

The transition of the above system can be written as

$$\Phi(t,0) = K(t)\, e^{Bt},$$

where $K(t) = K(t+T)$ and $K(0) = I$, $B = \frac{1}{T}\ln \Phi(T,0)$.

proof. Since $\dot{\Phi}(t,0) = A(t)\,\Phi(t,0)$, $\Phi(0,0) = I$.

$\dot{\Phi}(t+T,T) = A(t+T)\,\Phi(t+T,T) = A(t)\,\Phi(t+T,T)$. same init values

and $\Phi(0+T,T) = I$.

same dynamics

$\Rightarrow \Phi(t+T,T) = \Phi(t,0)$

$K(t+T) = \Phi(t+T,0)\, e^{-B(t+T)} = \Phi(t+T,0)\, e^{-BT} e^{-Bt}$

$\boxed{B = \frac{1}{T}\ln\Phi(T,0) \Rightarrow \Phi(T,0) = e^{BT}}$

$= \Phi(t+T,0)\, \Phi^{-1}(T,0)\, e^{-Bt} = \Phi(t+T,T)\cdot\cancel{\Phi(T,0)}\,\cancel{\Phi^{-1}(T,0)}\, e^{-Bt}$

$= \Phi(t+T,T)\, e^{-Bt} = \Phi(t,0)\, e^{-Bt} = K(t)$

The stability of γ is determined by the eigenvalues of e^{BT}.
$\hookrightarrow$ Floquet multipliers.

If $v \in \mathbb{R}^n$ is tangent to $\phi(0)$, then v is the eigenvector corresponding to the Floquet multiplier 1. The rest of the eigenvalues, if none is on the unit circle, determine the stability of γ.

Center manifold theory

Consider $\dot{x} = f(x)$, $x \in \mathbb{R}^n$, $f \in C^2$ in $B_r(0) \in \mathbb{R}^n$, $f(0) = 0$. (*)

Recall that we can determine the stability of the system via linearization

$$A = \left.\frac{\partial f(x)}{\partial x}\right|_{x=0}$$

- $x = 0$ is asymptotically stable if $Re(\lambda_i(A)) < 0, \ \forall i$
- $x = 0$ is unstable if $Re(\lambda_i(A)) > 0$ for some i.

But what happens when $Re(\lambda_i(A)) = 0$?

A k-dimensional manifold in $\mathbb{R}^n$ has its own rigorous definition. To ease the mathematical details, it is sufficient to think of it as the solution to $\eta(x) = 0$, $\eta: \mathbb{R}^n \mapsto \mathbb{R}^{n-k}$ is sufficiently smooth.

(*) can be rewritten as:

$$\dot{x} = \underbrace{\left.\frac{\partial f(x)}{\partial x}\right|_{x=0}}_{A} x + \underbrace{\left[f(x) - \underbrace{\left.\frac{\partial f(x)}{\partial x}\right|_{x=0}}_{A} x\right]}_{\tilde{f}(x)} \qquad (**)$$

$\tilde{f}(x) \in C^2$ and $\tilde{f}(0) = 0$, $\left.\frac{\partial \tilde{f}(x)}{\partial x}\right|_{x=0} = 0$

Our interest: when linearization "fails," i.e, $Re(\lambda_i(A)) \leq 0$.

Do a coordinate transformation:

$\begin{bmatrix} y \\ z \end{bmatrix} = Tx$, $y \in \mathbb{R}^k$, $z \in \mathbb{R}^m$. $TAT^{-1} = \begin{bmatrix} A_1 & \\ & A_2 \end{bmatrix}$, A_1: zero real parts, A_2: negative real parts

We can transform (**) into:

$$(***) \quad \begin{aligned} \dot{y} &= A_1 y + g_1(y,z) \\ \dot{z} &= A_2 z + g_2(y,z) \end{aligned}$$

$g_i(0,0) = 0$, $\left.\frac{\partial g_i(y,z)}{\partial y}\right|_{\substack{y=0\\z=0}} = 0$, $\left.\frac{\partial g_i(y,z)}{\partial z}\right|_{\substack{y=0\\z=0}} = 0$, $i = 1, 2$

If $z = h(y)$ is an invariant manifold for (***) and $h(\cdot)$ is smooth, then it is called a center manifold if $h(0) = 0$, $\left.\frac{\partial h(y)}{\partial y}\right|_{y=0} = 0$

Thm (Existence). There exists a constant $\delta > 0$ and $h(\cdot) \in C^1$, defined on $\|y\| < \delta$, such that $z = h(y)$ is a center manifold for (***)

Note that a center manifold is invariant. If $(y(0), z(0))$ starts in $z = h(y)$, it holds that $z(t) = h(y(t))$ $\forall t \geq 0$.

Model reduction.

It is sufficient to consider $\dot{y} = A_1 y + g_1(y, h(y))$

If $z(0) \neq h(y(0))$, then consider the system by change of variable: $w = z - h(y)$,

$$\begin{cases} \dot{y} = A_1 y + g_1(y, w + h(y)) \\ \dot{w} = \dot{z} - \frac{\partial h(y)}{\partial y} \cdot \dot{y} = A_2 z + g_2(y, z) - \frac{\partial h(y)}{\partial y} \cdot (A_1 y + g_1(y, z)) \end{cases}$$

$$= A_2(w + h(y)) + g_2(y, w + h(y)) - \frac{\partial h(y)}{\partial y}[A_1 y + g_1(y, w + h(y))]$$

Note that, the "motion" on the manifold is described by

$w(t) \equiv 0 \Rightarrow \dot{w}(t) \equiv 0$

$$0 = A_2 h(y) + g_2(y, h(y)) - \frac{\partial h(y)}{\partial y}[A_1 y + g_1(y, h(y))]$$

(Center manifold described by PDE)

$$\Rightarrow \dot{y} = A_1 y + g_1(y, w + h(y)) + \underbrace{g_1(y, h(y)) - g_1(y, h(y))}_{\text{"add \& subtract"}}$$

$$= A_1 y + g_1(y, h(y)) + \underbrace{N_1(y, w)}_{g_1(y, w+h(y)) - g_1(y, h(y))}$$

$$\dot{w} = A_2(w + h(y)) + g_2(y, w + h(y)) - \frac{\partial h(y)}{\partial y}[A_1 y + g_1(y, w + h(y))]$$

$$- \underbrace{[A_2 h(y) + g_2(y, h(y)) - \frac{\partial h(y)}{\partial y}[A_1 y + g_1(y, h(y))]]}_{=0}$$

$$= A_2 w + N_2(y, w)$$

$$N_2(y,w) = g_2(y, w + h(y)) - g_2(y, h(y)) - \frac{\partial h(y)}{\partial y}\underbrace{[g_1(y, w + h(y)) - g_1(y, h(y))]}_{N_1(y, w)}$$

$N_1, N_2 \in C^2$, $N_i(y,0) = 0$, $\frac{\partial N_i}{\partial w}\Big|_{\substack{y=0 \\ w=0}} = 0$, $i=1,2$.

$\Rightarrow \|N_i(y,w)\| \le k_i \|w\|$, $i=1,2$. for $\left\|\begin{bmatrix} y \\ w \end{bmatrix}\right\| < \rho$. (mean value thm)

where k_1, k_2 can be made arbitrarily small by choosing ρ small enough.

This suggest that the stability is governed by $\underline{\dot{y} = A_1 y + g_1(y, h(y))}$ since A_2 is Hurwitz.

Thm (Reduction Principle)

If the origin $y=0$ of $\dot{y} = A_1 y + g_1(y, h(y))$ is asymptotically stable, then the origin of $\begin{cases} \dot{y} = A_1 y + g_1(y,z) \\ \dot{z} = A_2 y + g_2(y,z) \end{cases}$ is asymptotically stable.

To use the Reduction principle, we need to find the center manifold $z = h(y)$. Recall that $z = h(y)$ is a solution to the PDE.

$$[M\, h](y) = \frac{\partial h}{\partial y}(y)\left[A_1 y + g_1(y, h(y))\right] - A_2 h(y) - g_2(y, h(y)) = 0$$

Thm. If $\phi(y) \in C^1$, and $\phi(0) = 0$, $\frac{\partial \phi}{\partial y}\Big|_{y=0} = 0$, and

$[M\phi](y) = O(\|y\|^q)$, for some $q > 1$, then for sufficiently small $\|y\|$,

$h(y) = \phi(y) + O(\|y\|^q)$,

The reduced system can be represented by $\dot{y} = A_1 y + g_1(y, \phi(y)) + O(\|y\|^q)$

Ex

$$\begin{aligned} \dot{x}_1 &= x_1 x_2^3 \\ \dot{x}_2 &= -x_2 - x_1^2 \end{aligned} \quad \Rightarrow \quad \text{Linearization around } x=0 \quad \dot{x} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} x, \quad \lambda_1 = 0 \quad \lambda_2 = -1$$

critical case.

$$\begin{aligned} \dot{x}_1 &= \overset{A_1}{0} \cdot x_1 + x_1 x_2^3 \\ \dot{x}_2 &= \underset{A_2}{-1} \cdot x_2 - x_1^2 \end{aligned}$$

Try $x_2 = -x_1^2$ on $[M\phi](x_1)$, we have:

$$[M\phi](x_1) = -2x_1\left[x_1(x_1^2)^3\right] \cancel{- x_1^2 + x_1^2} = 2x_1^8$$

So $h(x_1) = -x_1^2 + O(x_1^8)$ and on the center manifold, we have.

$$\dot{x}_1 = x_1(x_2 = h(x_1))^3 = x_1 \cdot (-x_1^2 + O(x_1^8))^3 = -x_1^7 + O(x_1^{13})$$

$x_1 = 0$ is asymptotically stable, hence $x=0$ is asymptotically stable.

Singular Perturbation

Consider $\begin{cases} \dot{x} = f(t,x,z,\varepsilon), & x \in \mathbb{R}^n \quad (*) \\ \varepsilon\dot{z} = g(t,x,z,\varepsilon), & z \in \mathbb{R}^n \end{cases}$ $f, g \in C^1$, $\varepsilon > 0$ is "small"

Set $\varepsilon = 0$, the second part would degenerate into an algebraic equation $0 = g(t,\bar{x},\bar{z},0)$. We call $\{\bar{x},\bar{z} \mid g(t,\bar{x},\bar{z},0)=0\}$ "domain of interest".

If domain of interest has $p \geq 1$ distinct (isolated) real roots

$$\bar{z} = \phi_i(t,\bar{x}), \quad i = 1, \cdots p,$$

$(*)$ is called "in standard form".

$\Rightarrow$ For each root, we have $\dot{\bar{x}} = f(t,\bar{x},\phi(t,\bar{x}),\varepsilon)$ ("Reduced model") $(**)$

<u>Ex</u> "High gain" amplifier

$$\dot{x} = z$$
$$\dot{z} = -kx - z - k\tan(z) + ku,$$

k is the high "control gain".

Set $\varepsilon = 1/k \Rightarrow \begin{array}{l} \dot{x} = z \\ \varepsilon\dot{z} = -x - \varepsilon z - \tan(z) + u \end{array} \Rightarrow$ set $\varepsilon = 0$ $\quad \bar{z} = \tan^{-1}(\bar{u} - \bar{x})$

$$\Rightarrow \dot{\bar{x}} = \tan^{-1}(\bar{u} - \bar{x})$$

! when ε is sufficiently small, $(*)$ has <u>two-time-scale behaviours</u>, (slow & fast). Under suitable assumptions, the <u>slow response</u> is approximated by the "<u>reduced model</u>", while the discrepancy between the response of "reduced model" and the original model is the "<u>fast transient</u>".

Suppose $(*)$ starts at (x_0, z_0), $(**)$ start at $\bar{x}(t_0) = x_0$, but $\bar{z}(t_0) = \phi(t_0, x_0)$ <u>may be quite different from</u> z_0

$\Rightarrow \bar{z}(t)$ can not be a uniform approximation of $z(t)$.

"Best we can hope" — <u>$z(t) = \bar{z}(t) + O(\varepsilon)$</u>, $\forall t \in [t_1, T]$

$t_1 > t_0$ holds on some interval.

However, for $x(t)$ the approximation can be uniform, i.e.,

$$x(t) = \bar{x}(t) + O(\varepsilon), \quad \forall t \in [t_0, T]$$

change of variable $y = z - \underbrace{\phi(t,x)}_{\bar{z}}$

$$\dot{x} = f(t,x,y+\phi(t,x),\varepsilon),$$

$$(**)\quad \varepsilon\dot{y} = \varepsilon\dot{z} - \varepsilon\dot{\phi}(t,x) = g(t,x,y+\phi(t,x),\varepsilon) - \varepsilon\frac{\partial\phi}{\partial t} - \varepsilon\cdot\frac{\partial\phi}{\partial x}\cdot f(t,x,y+\phi(t,x),\varepsilon)$$

$$x(t_0) = \xi(\varepsilon), \quad y(t_0) = \eta(\varepsilon) - \phi(t_0,\xi(\varepsilon))$$

Domain of interest: $y=0$, we want to analyze $(**)$ in a different time-scale

$\varepsilon\frac{dy}{dt} = \frac{dy}{d\tau} \Rightarrow \frac{d\tau}{dt} = 1/\varepsilon$ and use $\tau=0$ as the "initial time-instant" at $t=t_0$

$$\Rightarrow \tau = \frac{t-t_0}{\varepsilon} \Rightarrow \frac{dy}{d\tau} = g(t,x,y+\phi(t,x),\varepsilon) - \varepsilon\frac{\partial\phi}{\partial t} - \varepsilon\frac{\partial\phi}{\partial x}f(t,x,y+\phi(t,x),\varepsilon)$$

$$y(0) = \eta(\varepsilon) - \phi(t_0,\xi(\varepsilon))$$

x and t would change very slowly since $t = t_0 + \varepsilon\tau$, $x = x(t_0+\varepsilon\tau;\varepsilon)$

Set $\varepsilon=0$ to "freeze" $t=t_0$, $x=\xi_0$. $\underline{\text{(Boundary-layer system)}}\ (***)$

$$\Rightarrow \frac{dy}{d\tau} = g(t_0,\xi_0,y+\phi(t_0,\xi_0),0), \quad y(0) = \eta(0) - \phi(t_0,\xi_0) := \eta_0 - \phi(t_0,\xi_0)$$

with equilibrium $y=0$. If $y=0$ is asymptotically stable and $y(0)$ is in the domain of attraction, intuitively, $(***)$ would reach an $O(\varepsilon)$ neighbourhood of the origin. $z(t) = \underset{\text{slow}}{\bar{z}(t)} + \underset{\text{fast}}{\underline{y\left(\frac{t-t_0}{\varepsilon}\right)}} + O(\varepsilon)$

Thm (Tikhonov)

Suppose i) the equilibrium $y=0$ of boundary-layer system is uniformly asymptotically stable in ξ_0 and t_0, $z_0 - \bar{z}(t_0) = y(0)$ belongs to its domain of attraction,

ii) $\mathrm{Re}\,\lambda\left\{\frac{\partial g}{\partial z}(t,\bar{x}(t),\bar{z}(t),0)\right\} \le -c < 0$, $\forall t\in[t_0,T]$.

then $x(t) = \bar{x}(t) + O(\varepsilon)$, $z(t) = \bar{z}(t) + y\left(\frac{t-t_0}{\varepsilon}\right) + O(\varepsilon)$ holds $\forall t\in[t_0,T]$,

while $z(t) = \bar{z}(t) + O(\varepsilon)$ holds $\forall t\in[t_1,T]$.

: "Thickness of the boundary layer" $t_1 - t_0$ can be made arbitrarily small by choosing sufficiently small ε.