Nonlinear Control Theory

Lecture 7. Controllability

Last time

- Stability of invariant set (Poincaré map, Floquet's theory)
- Center manifold theory
- Singular perturbation.

Today

Controllability (nonlinear affine systems)

Let us first consider the linear case:

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \; u \in \mathbb{R}^m. \quad (*)$$

Def: $S \subseteq \mathbb{R}^n$ is called a controlled invariant subspace of $(*)$ if there exists a feedback control $u = Fx$ such that S is an invariant set under $\dot{x} = (A+BF)x$.

Def S is (A, B)-invariant (controlled invariant) subspace if there exists a matrix F, such that $(A+BF)S \subseteq S$ ($x \in S \Rightarrow (A+BF)x \in S$)

Such F is called a friend of S.

Idea:

$$x(t) = e^{(A+BF)t} x_0 = \left(I + (A+BF)t + \frac{1}{2}(A+BF)^2 t^2 + \cdots\right) x_0$$

If $x_0 \in S$, $(A+BF)x_0 \in S$, $\Rightarrow$ $x(t) \in S$. $\forall t \geq 0$ $\Rightarrow$ sufficiency.

On the other hand, suppose there exists a point $x_0 \in S$, such that $(A+BF)x_0 \notin S$,

$$x(t; x_0) = x_0 + t\underbrace{(A+BF)x_0}_{\notin S} + o(t^2) \notin S, \text{ for sufficiently small } t.$$

$\Rightarrow$ it is also necessary.

Def. A control system is called controllable if for any two points $x_1, x_2 \in \mathbb{R}^n$, there exists a finite time T and an admissible control $u \in \mathbb{R}^m$, such that $x(T, 0, u; x_1) = x_2$, where $x(t, t_0, u; x_0)$ denotes the solution at time t with initial condition x_0, initial time t_0. and control $u(\cdot)$

For linear systems, $x_1 = X(T) = e^{AT} x_0 + \int_0^T e^{A(T-\tau)} B u(\tau)\, d\tau$.

$$\Rightarrow \underbrace{\int_0^T e^{A(T-\tau)} B u(\tau)\, d\tau}_{\mathcal{L}(u),\ \mathcal{L}: \mathcal{U} \mapsto \mathbb{R}^n \text{ linear mapping}} = \underbrace{x_1 - e^{AT} x_0}_{d} \Rightarrow \mathcal{L}(u) = d.$$

"controllability" $\Leftrightarrow$ the linear mapping $\mathcal{L}(u)$ is "on-to" for $t \in [0, T]$

on-to　　　　　　　　not on-to

$\Leftrightarrow$ the rows of $e^{A(T-\tau)} B$ are linearly independent.

$$\downarrow$$

$$\sum_{j=0}^{\infty} \frac{(T-\tau)^j}{j!} A^j B \Rightarrow$$ The linear system is controllable iff $\mathcal{R} = \mathrm{Im}(B\ AB\ \cdots\ A^{n-1}B)$ has dimension n.

Denote $\langle A | S \rangle$ as the smallest $\underbrace{A\text{-invariant subspace}}_{AS \subseteq S}$ that contains S,

$\langle A | \mathrm{Im}\, B \rangle = \mathcal{R} = \mathrm{Im}(B, AB, \cdots, A^{n-1}B)$.

Q: Can we extend this idea to the nonlinear case $\underbrace{\dot{x} = f(x) + g(x) \cdot u}_{\text{nonlinear affine systems}}$? (**)

Namely, $\mathrm{span}\{g_1(x), g_2(x), \cdots, g_m(x)\} \approx \mathrm{Im}\, B$,

Proposition. Consider system (**) and x_0 where $f(x_0) = 0$. If the linearization at x_0 and $u = 0$

$$\dot{z} = \left.\frac{\partial f}{\partial x}\right|_{x = x_0} z + g(x_0) u$$

then, the set of points that can be reached from x_0 in any finite time contains a neighbourhood of x_0.

Ex. Consider

$$\begin{cases} \dot{x} = \cos\theta \cdot u_1 \\ \dot{y} = \sin\theta \cdot u_1 \\ \dot{\theta} = u_2 \end{cases} \xrightarrow{\text{linearize at } (x_0, y_0, \theta_0)} \begin{array}{l} \dot{x} = \cos\theta_0\, u_1 \\ \dot{y} = \sin\theta_0\, u_1 \\ \dot{\theta} = u_2 \end{array} \longrightarrow \text{clearly not controllable!}$$

But you can drive a car anywhere you like.

The question is: whether the nonlinear system itself is controllable?

Before proceed, let's cover some knowledge about differential geometry.

We have covered the "definition" of manifold briefly in the previous lecture.

Def (Manifold on $\mathbb{R}^n$)

Suppose $\mathcal{N}$ is an open set in $\mathbb{R}^n$. The set M defined as

$$M = \{ x \in \mathcal{N} : \lambda_i(x) = 0, i = 1, \cdots, n-m \}$$

where λ_i are smooth functions.

If $\operatorname{rank} \begin{bmatrix} \frac{\partial \lambda_1}{\partial x} \\ \vdots \\ \frac{\partial \lambda_{n-m}}{\partial x} \end{bmatrix} = n-m \quad \forall x \in M$, then M is a (hyper)surface (which is a smooth manifold of dimension $n-m$).

Take a vector $b \in \mathbb{R}^n$ and smooth function $\lambda : \mathbb{R}^n \mapsto \mathbb{R}$, then at any point $x \in \mathbb{R}^n$, the rate of change of $\lambda(x)$ along the direction of b is:

$$\mathcal{L}_b \lambda := \lim_{\varepsilon \to 0} \frac{\lambda(x + \varepsilon b) - \lambda(x)}{\varepsilon}$$

"since ε is small"

Do Taylor expansion to $\lambda(x+\varepsilon b)$ at x, we have:

$$\mathcal{L}_b \lambda = \lim_{\varepsilon \to 0} \frac{\lambda(x) + \varepsilon \cdot b \frac{\partial \lambda}{\partial x} - \lambda(x)}{\varepsilon} = b \frac{\partial \lambda}{\partial x}$$

$$= \underbrace{\left(\sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} \right)}_{b} \lambda$$

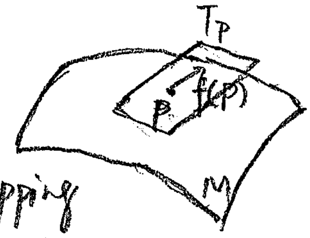
"operator form".

tangent vector

We see $\{ \frac{\partial}{\partial x_i} \}$ $i = 1, \cdots, n$ as a basis of the tangent space to $\boxed{\mathbb{R}^n}$

it is also a manifold itself.

Tangent space to a general manifold M at a point P (denoted by $T_P M$) can be defined similarly. The precise definition will not be covered in this course.



Def (Vector field).

A vector field f on a smooth manifold M is a mapping assigning to each point $p \in M$ a tangent vector $f(p) \in T_p M$.

A vector field is smooth over $\mathbb{R}^n$ (where $M = \mathbb{R}^n$) if there exists n real valued smooth functions $f_1, \cdots, f_n$ defined on $\mathbb{R}^n$, such that

$$f(q) = \sum_{i=1}^{n} f_i(q) \frac{\partial}{\partial x_i}, \quad \forall q \in \mathbb{R}^n$$

where $x_1, \cdots x_n$ form a basis for $\mathbb{R}^n$.

We see the solution of ODE $x(t; f, p)$ as "flow" and denote as
(vector field) (initial value)

$\gamma^f_t(p)$. It holds that $\gamma^f_{t_1} \circ \gamma^f_{t_2} = \gamma^f_{t_1+t_2}$, $\gamma^f_{-t} = (\gamma^f_t)^{-1}$, $\gamma^f_0 = id$.

(Compare with the state transition matrix). It also holds that:

$$\left.\frac{\partial}{\partial t}\right|_{t=0} \gamma^f_t(p) = f(\gamma^f_t(p))\Big|_{t=0} = f(p)$$

Def (Lie bracket in $\mathbb{R}^n$)

For two vector fields f and g on $M \subset \mathbb{R}^n$, the Lie bracket is a third vector field defined by $[f, g](x) = \frac{\partial g}{\partial x} \cdot f(x) - \frac{\partial f}{\partial x} \cdot g(x)$

Geometric interpretation

Lemma For every $x \in \mathbb{R}^n$, $\gamma^g_{-t} \circ \gamma^f_{-t} \circ \gamma^g_t \circ \gamma^f_t(p) = p + t^2 [f, g](p) + o(t^3)$,
for t that tends to zero.

proof: It's enough we do Taylor expansion for each flow to the order 3.

$$\begin{aligned}
\gamma^f_t(p) &= p + \left.\frac{\partial}{\partial t}\right|_{t=0} \gamma^f_t(p)\, t + \frac{t^2}{2} \left.\frac{\partial^2}{\partial t^2}\right|_{t=0} \gamma^f_t(p) + o(t^3) \\
&= p + t \cdot f(\gamma^f_t(p))\Big|_{t=0} + \frac{t^2}{2} \frac{\partial}{\partial t}\left[f(\gamma^f_t(p))\right]\Big|_{t=0} + o(t^3) \\
&= p + t f(p) + \frac{t^2}{2} \cdot Df(\gamma^f_t(p)) . f(\gamma^f_t(p))\Big|_{t=0} + o(t^3) \\
&= p + t f(p) + \frac{t^2}{2} Df(p) f(p) + o(t^3).
\end{aligned}$$

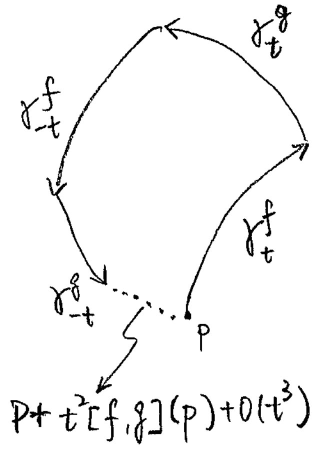
D: denotes the Jacobian

$$f = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} \quad Df = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

$$\begin{aligned}
\gamma^g_t \circ \gamma^f_t(p) &= \gamma^f_t(p) + t \cdot g(\gamma^f_t(p)) + \frac{t^2}{2} Dg(\gamma^f_t(p)) g(\gamma^f_t(p)) + o(t^3) \\
&= p + t f(p) + \frac{t^2}{2} Df(p) f(p) + o(t^3) \\
&\quad + t \cdot \left[g(p) + t \cdot Dg(\gamma^f_t(p)) f(\gamma^f_t(p))\Big|_{t=0} + o(t^2) \right] \\
&\quad + \frac{t^2}{2} \left[Dg(p) g(p) + o(t) \right] \\
&= p + t[f(p) + g(p)] + \frac{t^2}{2} Df(p) f(p) + t^2 Dg(p) f(p) + \frac{t^2}{2} Dg(p) g(p) + o(t^3).
\end{aligned}$$

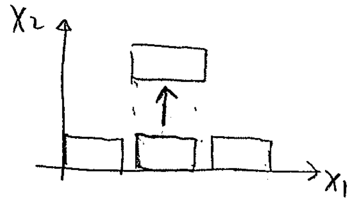
Similarly, $\gamma^f_{-t} \circ \gamma^g_t \circ \gamma^f_t(p) = x + t \cdot g(x) + [f, g](p) t^2 + o(t^3)$

and $\gamma^g_{-t} \circ \gamma^f_{-t} \circ \gamma^g_t \circ \gamma^f_t(p) = p + t^2 [f, g](p) + o(t^3)$



! The Lemma states the fact that if $[f,g](p) \notin \text{span}\{f(x), g(x)\}$, then it is possible, by alternating between the flow of f and g, to attain the points that can not be reached with the flow of linear combinations of f and g.

Ex (Car parking)



Dynamics: $\dot{x}_1 = V\cos\theta$ $\qquad$ $\dot{x}_1 = V\cos\theta$

$\dot{x}_2 = V\sin\theta$ $\qquad$ $\dot{x}_2 = V\sin\theta$

$\dot{x}_3 = \omega$ $\qquad$ $\dot{x}_3 = -\omega$

f vector-field $\qquad$ g vector-field.

$$f : \begin{bmatrix} V\cos x_3 \\ V\sin x_3 \\ \omega \end{bmatrix} \qquad g : \begin{bmatrix} V\cos x_3 \\ V\sin x_3 \\ -\omega \end{bmatrix}$$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 0 & -V\sin x_3 \\ 0 & 0 & V\cos x_3 \\ 0 & 0 & 0 \end{bmatrix} \qquad \frac{\partial g}{\partial x} = \begin{bmatrix} 0 & 0 & -V\sin x_3 \\ 0 & 0 & V\cos x_3 \\ 0 & 0 & 0 \end{bmatrix}$$

Lie bracket:

$$[f,g] = \frac{\partial g}{\partial x}\cdot f - \frac{\partial f}{\partial x}\cdot g$$

$$= \begin{bmatrix} 0 & 0 & -V\sin x_3 \\ 0 & 0 & V\cos x_3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} V\cos x_3 \\ V\sin x_3 \\ \omega \end{bmatrix} - \begin{bmatrix} 0 & 0 & -V\sin x_3 \\ 0 & 0 & V\cos x_3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} V\cos x_3 \\ V\sin x_3 \\ -\omega \end{bmatrix}$$

$$= \begin{bmatrix} -\omega V\sin x_3 \\ \omega V\cos x_3 \\ 0 \end{bmatrix} - \begin{bmatrix} \omega V\sin x_3 \\ -\omega V\cos x_3 \\ 0 \end{bmatrix} = 2\omega V\begin{bmatrix} -\sin x_3 \\ \cos x_3 \\ 0 \end{bmatrix}$$

$$x_3 = 0 \Rightarrow [f,g]\Big|_{x_3=0} = 2\omega V\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

$V > 0$, $\omega > 0$.

γ_t^f : LF., γ_t^g : RF, γ_{-t}^f : RB, γ_{-t}^g : LB.

Denote $\text{ad}_f g = [f,g]$, $\text{ad}_f^0 g = g$, $\text{ad}_f^k g = [f, \text{ad}_f^{k-1} g]$

Def. A distribution Δ on a manifold M is a map which assigns to each $p \in M$ a vector subspace $\Delta(p)$ of the tangent space T_pM.

Δ is called ~~a~~ smooth if for each $p \in M$, there exists a neighbourhood U of p and ~~a~~ a set of smooth vector fields f_i, $i \in I$, such that

$$\Delta(q) = \text{span}\{f_i(q)\}, \ \forall q \in U.$$

a subset of a vector space
$x, y \in S$, then $\alpha x + \beta y \in S$.

Throughout the course, we always assume the distribution is smooth and the index set I is finite.

Def. A distribution is called nonsingular if for each $p \in M$, $\dim(\Delta(p))$ is the same, i.e., $\{f_i(p)\}$ are linearly independent $\forall p \in M$.

Def. A distribution Δ is called involutive if

$$f \in \Delta,\ g \in \Delta \Rightarrow [f, g] \in \Delta.$$

Now, consider the nonlinear affine system $\dot{x} = f(x) + g(x)u$.

$x \in N \subset \mathbb{R}^n$. ($N$: manifold.)

where $g(x) = [g_1(x), g_2(x), \cdots g_m(x)]$

The dynamics is actually: $\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i$

Def. A distribution $\Delta(x)$ is said to be invariant under vector field $f(x)$, if $[f, k](x) \in \Delta(X)$, $\forall k \in \Delta(X)$.

Def (Strong accessibility distribution R_c)

R_c is the smallest distribution which contains $\mathrm{span}\{g_1, \cdots, g_m\}$ and is invariant under vector fields $f, g_1, \cdots, g_m$ and is denoted by

$$R_c(X) = \langle f, g_1, \cdots, g_m \mid \mathrm{span}\{g_1, \cdots, g_m\} \rangle$$

For linear systems, $f = Ax$, $g_i = b_i$ ($B = [b_1, \cdots, b_m]$)

The strong accessibility distribution is $\langle AX; b_1, \cdots b_m \mid \mathrm{span}\{b_1, \cdots b_m\} \rangle$

We get b_i-invariance for free, since $[b_i, b_j] = 0$, $\forall i, j = 1, \cdots m$.

$$[Ax, b_i] = \frac{\partial b_i}{\partial x} \cdot Ax - \frac{\partial Ax}{\partial x} \cdot b_i = -Ab_i$$

$$\Rightarrow R_c(X) = \langle A \mid \mathrm{span}\{b_1, \cdots b_m\} \rangle = \langle A \mid \mathrm{Im}\, B \rangle$$

Controllable subspace of linear systems.

For nonlinear systems, it is in general very difficult to determine the controllability except for some special cases. Thus, it is very useful to study the so-called accessibility.

Proposition If at a point x_0, $\dim(R_c(x_0)) = n$, then the system is locally strongly accessible from x_0. Namely, for any neighbourhood of x_0, the set of reachable points at time T contains a non-empty open set for any $T > 0$.

Ex 1. Consider the angular motion of a space-craft. Here we assume only two controls (two pairs of boosters) are available.
The model of angular velocities around the three main axes is:

$$\dot{x}_1 = \frac{a_2 - a_3}{a_1} x_2 x_3$$

$$\dot{x}_2 = \frac{a_3 - a_1}{a_2} x_1 x_3 + u_1 \qquad , a_1, a_2, a_3 > 0$$

$$\dot{x}_3 = \frac{a_1 - a_2}{a_3} x_2 x_1 + u_2$$

$$f(x) = \begin{bmatrix} \alpha x_2 x_3 \\ \beta x_1 x_3 \\ \gamma x_1 x_2 \end{bmatrix}, \quad g_1(x) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad g_2(x) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \alpha = \frac{a_2 - a_3}{a_1}, \quad \beta = \frac{a_3 - a_1}{a_2}, \quad \gamma = \frac{a_1 - a_2}{a_3}$$

step 1

$$R_0(x) = \text{span}\{g_1(x), g_2(x)\} = \text{span}\{e_2, e_3\}$$

step 2 Lie brackets:

$$g_3(x) := [f, g_1] = \frac{\partial g_1}{\partial x} \cdot f - \frac{\partial f}{\partial x} g_1 = -\begin{bmatrix} 0 & \alpha x_3 & \alpha x_2 \\ \beta & 0 & \beta x_1 \\ \gamma x_2 & \gamma x_1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = -\begin{bmatrix} \alpha x_3 \\ 0 \\ \gamma x_1 \end{bmatrix}$$

$$g_4(x) := [f, g_2] = \frac{\partial g_2}{\partial x} \cdot f - \frac{\partial f}{\partial x} g_2 = -\begin{bmatrix} \alpha x_2 \\ \beta x_1 \\ 0 \end{bmatrix}, \quad [g_1, g_2] = 0.$$

$$R_1(x) = \text{span}\{g_i(x), i = 1, \cdots 4\}$$

step 3. If $\alpha = 0$ (i.e. $a_2 = a_3$), then $R_1(x) = R_0(x)$, $R_C(x) = R_0(x) = \text{span}\{e_2, e_3\}$

If $\alpha \neq 0$, then $R_1(x) \neq R_0(x)$ and $\dim R_1(x) = 2 < 3$ for $x_2 = x_3 = 0$.
Hence return to step 2

step 2-2 $R_2(x) = R_1(x) + \text{span}\{[f, g_i], [g_i, g_j], i = 1, 2, 3, 4\}$.

Since $[g_1, g_4] = \frac{\partial g_4}{\partial x} \cdot g_1 = \begin{bmatrix} 0 & -\alpha & 0 \\ -\beta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\alpha \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow R_2(x) = \mathbb{R}^3$. $\dim(R_2(x)) = 3 \ \forall x \in \mathbb{R}^3$,

$\Rightarrow R_c(x) = R_2(x) = \mathbb{R}^3 \Rightarrow$ The system is locally strongly accessible from any point in $\mathbb{R}^3$

Thm (Chow)

If $f=0$, then $\dim(R_c(x)) = n$, $\forall x \in N$ implies the system is controllable.

Ex). Unicycle model

$$\dot{x}_1 = u_1 \cos x_3$$
$$\dot{x}_2 = u_1 \sin x_3 \quad \Rightarrow \quad \dot{x} = \begin{bmatrix} \cos x_3 & 0 \\ \sin x_3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \Rightarrow g_1(x) = \begin{bmatrix} \cos x_3 \\ \sin x_3 \\ 0 \end{bmatrix}$$
$$\dot{x}_3 = u_2 \qquad\qquad g_2(x) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$R_0(x) = \text{span}\{g_1(x), g_2(x)\}$$

$$g_3(x) := [g_1, g_2] = \frac{\partial g_2}{\partial x} \cdot g_1 - \frac{\partial g_1}{\partial x} \cdot g_2 = -\begin{bmatrix} 0 & 0 & -\sin x_3 \\ 0 & 0 & \cos x_3 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\sin x_3 \\ \cos x_3 \\ 0 \end{bmatrix}.$$

$R_1(x) = R_0(x) + \text{span}\{[g_1, g_2]\} = \mathbb{R}^3 \Rightarrow$ controllable.